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On Weak Permutability between Groups*

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1. INTRODUCTION

The permutability of two subgroups H and K of a group G is equivalent to the existence of certain functions $\alpha: H \times K \to K$, $\beta: H \times K \to H$ such that the permutability relations

$$hk = \alpha(h, k)\,\beta(h, k)$$

hold for all $h \in H$, $k \in K$. Our purpose in this paper is to investigate a weakened form of permutability which assumes fewer relations between H and K.

The degree of permutability chosen to be studied is well illustrated by a beautiful result of H. S. M. Coxeter [3]. This result states that given integers $m > 2$, $p_i > 1$ ($i = 1,\ldots, [m/2]$), the group

$$\mathscr{C}(m; p_1,\ldots, p_{[m/2]}) = \left\{a_1, a_2 \mid a_1{}^m = a_2{}^m = (a_1{}^j a_2{}^j)^{p_j} = 1, \text{ for } 1 \leqslant j \leqslant \left[\frac{m}{2}\right]\right\}$$

is finite only if $p_j = 2$ for every $j \leqslant \frac{1}{2}m$. Furthermore, the group $\mathscr{C}(m; 2,\ldots, 2)$, which is abbreviated by $\mathscr{C}(m; 2)$; is a semi-direct product of an elementary abelian 2-group of rank $m - 1$ by a cyclic group of order m. Thus the group $\mathscr{C}(m; 2)$ is generated by two isomorphic proper subgroups H_1 $(=\langle a_1\rangle)$, H_2 $(=\langle a_2\rangle)$ such that

$$H_1H_2 \cap H_2H_1 \supset \{h\gamma(h) \mid h \in H\},$$

where the map $\gamma: H_1 \to H_2$, defined by $\gamma(a_1{}^i) = a_2{}^i$, $0 \leqslant i < m$, is a bijection.

Even though H_1 and H_2 are not permutable inside $\mathscr{C}(m; 2)$; sufficient permutability relations still hold between them to ensure the finiteness of $\mathscr{C}(m; 2)$. In addition, these relations make it possible to characterize the group in question.

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Before proceeding with the description of the main results, let us introduce some necessary notation.

Let H and K be groups and R be a subset of the free product $H * K$. Then

$$\{H, K \mid \omega = 1, \forall \omega \in R\} \qquad \text{or} \qquad \langle H, K \mid R\rangle$$

stand for the quotient group of $H * K$ by the normal closure of $\langle \omega \mid \omega \in R\rangle$ in $H * K$.

When H and K are isomorphic groups and ψ: $H \to K$ is the isomorphism which holds between them, it is convenient to replace the symbol K by H^ψ, and denote the elements $\psi(h)$ of K by h^ψ. Note that $\langle H, K \mid R\rangle$ is not affected by the choice of the isomorphism ψ. This being the case, the usage of ψ for indicating different isomorphisms, as in

$$\langle H_1, H_1^\psi \mid R_1\rangle, \qquad \langle H_2, H_2^\psi \mid R_2\rangle,$$

where H_1 and H_2 are different groups, should not cause any confusion.

The rest of the notation and terminology is standard.

This paper is divided into four sections.

In Section 2, we present four general examples of groups where weak permutability holds. Of particular interest is

$$\mathscr{C}(H; 2) = \{H, H^\psi \mid (h^{-1}h^\psi)^2 = 1, \forall h \in H\},$$

which generalizes the $\mathscr{C}(m; 2)$ group. Here we prove

THEOREM A. *The group $\mathscr{C}(H; 2)$ is isomorphic to the semi-direct product of the augmentation ideal $\mathscr{A}_{\mathbb{Z}_2}(H)$ of the group ring $\mathbb{Z}_2(H)$, by H under the natural action.*

Observe that when H is a finite group of order n, then $|\mathscr{C}(H; 2)| = 2^{n-1}n$.

In Section 3, we provide a finiteness criterion for a group generated by two finite subgroups. A particular case is

THEOREM B. *Let G be a group and let H and K be nontrivial subgroups of G. Suppose that G is generated by H and K. Furthermore, suppose*

$$|H| = |K| = n,$$

and

$$HK \cap KH \supseteq \{h\gamma(h) \mid h \in H\},$$

where $\gamma\colon H \to K$ is a bijection and $\gamma(1) = 1$. Then, G is a finite group and

$$|G| < e^{n-1}n,$$

where e the base of the natural logarithms.

The fact that $\mathscr{C}(H; 2)$ satisfies the conditions of the theorem indicates that the upper bound obtained for $|G|$ is satisfactory. However, there are indications that $2^{n-1}n$ might very well be the least upper bound.

Section 4, which is the longest part of this paper, is a detailed study of the group

$$\mathscr{X}(H) = \{H, H^{\psi} \mid [h, h^{\psi}] = 1, \forall h \in H\}.$$

This group has $H \times H$ for a homomorphic image. $\mathscr{X}(H)$, just like $H \times H$ inherits many properties from H, although, in general, these two groups are different.

Some of the results of this section are gathered together in

THEOREM C. (i) *Let $\mathscr{P}$ be one of the following group theoretic properties: finite π-group (π: a set of primes), finite nilpotent, solvable of finite degree, perfect. Then,*

$$H \text{ is a } \mathscr{P}\text{-group} \Rightarrow \mathscr{X}(H) \text{ is a } \mathscr{P}\text{-group}.$$

(ii) *The group $\mathscr{X}(H)$ has a normal chain of subgroups such that one of its factors is isomorphic to the Schur Multiplier $M(H)$ of H.*

(iii) *Let H be a finite nontrivial group. Then,*

$$|H|^2 \, |H'| \, |M(H)| \leqslant |\mathscr{X}(H)| < |H| \, e^{|H|-1}.$$

In addition, the structure of $\mathscr{X}(H)$ is determined when H is a finite abelian group of odd order, and also when H is a perfect group which is not necessarily finite.

Note that subsections 4.3 and 4.4 contain a number of results on $M(H)$ and on covering groups of H (or stem extensions, according to Gruenberg [6, Sect. 9.9]). Some of these results provide new proofs to well-known theorems of I. Schur and W. Gaschutz.

This paper is basically self-contained.

2. Examples of Weakly Permutable Groups

2.1. *A Generalized Coxeter Group* $\mathscr{C}(H; 2)$

Let H and H^ψ be isomorphic groups through ψ, and define

$$[H, \psi] = \langle h^{-1}h^\psi \mid h \in H\rangle,$$
$$\mathscr{C}(H) = \{H, H^\psi \mid [H, \psi]' = 1\},$$
$$\mathscr{C}(H; 2) = \{H, H^\psi \mid (h^{-1}h^\psi)^2 = 1, \forall h \in H\}.$$

Also let $\mathscr{A}_{\mathbb{Z}}(H)$ and $\mathscr{A}_{\mathbb{Z}_2}(H)$ denote the augmentation ideals of the group rings $\mathbb{Z}(H)$ and $\mathbb{Z}_2(H)$, respectively.

Theorem 2.1.1. *Let G be the semi-direct product of $\mathscr{A}_{\mathbb{Z}}(H)$ by H under the natural action. Then, $\mathscr{C}(H)$ is a presentation of G.*

Proof. Let $\mathscr{E}$ be the ring of $\mathbb{Z}$-endomorphisms of the group $(\mathbb{Z}(H), +)$. Define

$$\phi\colon H \to \mathscr{E}$$

by $(x)\phi\colon h \to hx$ for all $h, x \in H$. Then, ϕ is an isomorphism of H into the multiplicative group of $\mathscr{E}$.

Let $\tilde{G} = (\mathbb{Z}(H) \times H)_\phi$ be the split extension of $\mathbb{Z}(H)$ by H with respect to ϕ, and let $G = \mathscr{A}_{\mathbb{Z}}(H) \times H$. Then, $\mathscr{A}_{\mathbb{Z}}(H) \times 1$ is a normal subgroup of $\tilde{G}$, and G a subgroup of $\tilde{G}$.

Define $c = (1, 1)$, $H_1 = \{0\} \times H$, $H_2 = H_1{}^c$. Then,

$$[h_1, c] = (-h^{-1} + 1, 1)$$

for any $h_1 = (0, h) \in H_1$. It follows from this fact that

$$[H_1, c] = \mathscr{A}_{\mathbb{Z}}(H) \times 1,$$
$$[H_1, c]' = 1,$$

and

$$G = \langle H_1, H_2\rangle.$$

Let $\xi\colon H \cup H^\psi \to G$ be defined by $\xi(h) = (0, h)$ and $\xi(h^\psi) = (0, h)^c$ for all $h \in H$. As G satisfies the conditions of $\mathscr{C}(H)$, ξ may be extended to a homomorphism from $\mathscr{C}(H)$ onto G. Since we have that

$$[H, \psi] \lhd \mathscr{C}(H), \qquad \mathscr{C}(H) = [H, \psi]H, \qquad \text{and} \qquad H \cap [H, \psi] = 1,$$

it is direct to verify that ξ is an isomorphism.

Now we prove Theorem A in the form of

COROLLARY 2.1.2. *Let $\bar{G}$ be the semi-direct product of $\mathscr{A}_{Z_2}(H)$ by H under the natural action. Then, $\mathscr{C}(H; 2)$ is a presentation of $\bar{G}$.*

Proof. Let $h_1, h_2 \in H$. Since we have

$$[h_1h_2, \psi] = [h_1, \psi]^{h_2}[h_2, \psi]$$

and

$$[h_1h_2, \psi]^2 = [h_1, \psi]^2 = [h_2, \psi]^2 = 1,$$

it follows that

$$[h_1, \psi]^{h_2} \text{ commutes with } [h_2, \psi],$$

and consequently,

$$[h_1, \psi] \text{ commutes with } [h_2, \psi].$$

In other words, $[H, \psi]' = 1$ holds. Clearly then, there exists an epimorphism $\zeta: G \to \bar{G}$ such that $\ker(\zeta) = 2\mathscr{A}_Z(H) \times 1$.

2.2. *Almost Permuting Groups*

Let

$$F \text{ be a right distributive near field,}$$

$$G = \{g \mid g: x \to (x)g = xm + a,\ m \in F^{\#},\ a \in F\}$$

be the group of linear substitutions of F. Also let

$$H = \{g \mid g: x \to xm,\ m \in F\},$$
$$c \text{ be the involution defined by } c: x \to -x + 1,$$
$$H_1 = H, \qquad H_2 = H^c.$$

As is well-known, G is sharply 2-transitive on the cosets of H in G. Thus, H_1 is a maximal subgroup of G, and as $H_1 \neq H_2$, $G = \langle H_1, H_2 \rangle$.

Let $h_i: x \to xm_i$, $1 \leqslant i \leqslant 4$, be elements of $H_1{}^{\#}$ such that

$$h_1{}^ch_2 = h_3h_4{}^c \text{ (equivalently, } h_1h_2{}^c = h_3{}^ch_4).$$

On applying $h_1{}^ch_2$ and $h_3h_4{}^c$ to x, we get

$$xm_1m_2 + (1 - m_1)\,m_2 = xm_3m_4 + (1 - m_4).$$

Thus,

$$m_3 = m_1 m_2 m_4^{-1}, \qquad m_4 = (m_1 - 1)m_2 + 1.$$

As $m_4 \neq 0$ and $m_1 \neq 1$, we have $m_2 \neq (1 - m_1)^{-1}$.

Define $\omega\colon F^{\#} \to F^{\#}$ by $\omega(1) = 1$ and $\omega(m) = (1 - m)^{-1}$ for $m \neq 1$. Also define $\hat{\omega}\colon H \to H$ by

$$\hat{\omega}(h)\colon x \to x\omega(m) \qquad \text{for} \quad h\colon x \to xm.$$

Then,

$$h_1 h_2{}^c \in H^c H$$

for all $h_1, h_2 \in H$, except for the pairs h_1 $(\neq 1)$, $h_2 = \hat{\omega}(h_1)$.

When F is finite of order $m > 2$, we may use a combinatorial theorem of P. Hall (see [7], p. 47) to conclude that there exist at least $(m - 3)!$ bijections $\gamma\colon H \to H^c$ such that $\gamma(1) = 1$, and $h\gamma(h) \in H^c H$, for all $h \in H$.

2.3. *Weakly Commuting Groups*, $T(\tilde{H})$

Let H be a group. Let $\tilde{H}$ be some covering group of H, in the sense that there exists Z a subgroup of $\tilde{H}$ such that

$$Z \subseteq Z(\tilde{H}) \cap \tilde{H}' \qquad \text{and} \qquad \tilde{H}/Z = H;$$

that is, $(Z \mid \tilde{H})$ is a stem extension of H. Define

$$\tilde{T} = \tilde{H} \times \tilde{H} \times \tilde{H},$$

$$\tilde{\psi} \in \operatorname{Aut}(\tilde{T}) \qquad \text{such that}$$

$$\tilde{\psi}\colon (a, b, c) \to (c, b, a) \qquad \text{for all } a, b, c \in \tilde{H},$$

$$\tilde{H}_1 = \{(a, a, 1) \mid a \in \tilde{H}\}, \qquad \tilde{H}_2 = \tilde{\psi}(\tilde{H}),$$

$$Z_1 = \{(z, z, 1) \mid z \in Z\}, \qquad Z_2 = \tilde{\psi}(Z_1),$$

$$\tilde{G} = \langle \tilde{H}_1, \tilde{H}_2 \rangle,$$

$$T(\tilde{H}) = G = \frac{\tilde{G}}{Z_1 Z_2},$$

$$H_1 = \frac{\tilde{H}_1 Z_2}{Z_1 Z_2}, \qquad H_2 = \frac{\tilde{H}_2 Z_1}{Z_1 Z_1}.$$

Then,

$$T(\tilde{H}) = \langle H_1, H_2 \rangle,$$

$$H_1 \cong H \cong H_2,$$

$$Z_1 Z_2(a, a, 1) \text{ commutes with } Z_1 Z_2(1, a, a) \text{ for every } a \in \tilde{H}.$$

We note that $\tilde{\psi}^2 = 1$ and so, $\tilde{G}$ and Z_1Z_2 are $\tilde{\psi}$-invariant. Hence, $\tilde{\psi}$ induces an automorphism ψ on $T(\tilde{H})$, such that

$$\psi: Z_1Z_2(a, a, 1) \leftrightarrow Z_1Z_2(1, a, a)$$

for all $a \in \tilde{H}$. Thus we have obtained

$$G = \langle H_1, H_1^{\psi}\rangle, \qquad H_1 \cong H,$$

and

$$[h_1, h_1^{\psi}] = 1 \qquad \text{for all } h_1 \in H_1 .$$

Remarks 2.3.1. Some of the following information concerning $T(\tilde{H})$ will be needed in Section 4.

(i) $T(\tilde{H})$ is a homorphic image of

$$\mathscr{X}(H) = \{H, H^{\psi} \mid [h, h^{\psi}] = 1, \text{ for all } h \in H\}.$$

(ii) Let $x \in \tilde{G}$. Then there exist $a_i \in \tilde{H}$, $1 \leqslant i \leqslant 2k$, such that

$$\begin{aligned} x &= (a_1, a_1, 1)(1, a_2, a_2) \cdots (1, a_{2k}, a_{2k}) \\ &= (a_1a_3 \cdots a_{2k-1}, a_1a_2 \cdots a_{2k}, a_2a_4 \cdots a_{2k}). \end{aligned}$$

(iii) Define $\tilde{D} = [\tilde{H}_1, \tilde{H}_2]$, $\tilde{L} = [\tilde{H}_1, \tilde{\psi}]$, subgroups of $\tilde{G}$, and let D, L be their respective images in G $(=T(\tilde{H}))$. Then,

$$\begin{gathered} \tilde{D}, \tilde{L} \lhd \tilde{G}, \\ \tilde{D} = 1 \times \tilde{H}' \times 1, \qquad \tilde{D} \cap Z_1Z_2 = 1, \\ \tilde{L} = \langle (a^{-1}, 1, a) \mid a \in \tilde{H}\rangle. \end{gathered}$$

Also,

$$\begin{aligned} \tilde{G} &= \tilde{L}\tilde{H}_i, \qquad \tilde{L} \cap \tilde{H}_i = 1, \\ G &= LH_i, \qquad L \cap H_i = 1, \end{aligned}$$

for $i = 1, 2$.

(iv) Given $x \in \tilde{G}$ and $y \in \tilde{H}$, let $\bar{x}$ denote the image of x in G and let $\bar{y} = Zy$ in $\tilde{H}/Z$ $(=H)$. Define

$$\begin{aligned} \alpha&: T(\tilde{H}) \rightarrow H \times H \\ \beta&: T(\tilde{H}) \rightarrow H \end{aligned}$$

by $\alpha(x) = (\bar{x}_1, \bar{x}_3)$, $\beta(\bar{x}) = \bar{x}_2$ for all $x = (x_1, x_2, x_3) \in \tilde{G}$. Then it follows that α and β are epimorphisms, such that

$$\ker(\alpha) = D, \qquad \ker(\beta) = L.$$

(v) The facts $\tilde{D} \supseteq 1 \times Z \times 1$ and $\tilde{D} \cap Z_1Z_2 = 1$, imply that

$$D \cap L = \overline{1 \times Z \times 1}.$$

(vi) Let $\tilde{H}$ be a perfect group. Then,

$$\tilde{D} = 1 \times \tilde{H} \times 1, \qquad \tilde{L} = \tilde{H} \times 1 \times \tilde{H}, \qquad \tilde{G} = \tilde{T},$$

and $T(\tilde{H}) = \tilde{T}/Z_1Z_2$.

2.4. $PSL(2, q)$

(a) Let $G = PSL(2, q)$, $q = p^m$, p: odd prime, $m \geqslant 1$. Let $H = GF(q)$, and let H^ψ be a field isomorphic to H. Define γ: $H \to H^\psi$ by $\gamma(\alpha) = (1/\alpha)^\psi$ for $\alpha \neq 0$, and $\gamma(0) = 0$. Then, by a theorem of Beetham [1], G has the presentation

$$\{H, H^\psi \mid (\alpha\gamma(\alpha))^2 = 1, \text{ for all } \alpha \in H\}.$$

(b) Let $G = SL(2, q)$, $q = 2^m$, $m \geqslant 2$. Then, as is well-known, G contains a dihedral subgroups $D = \langle a, b \rangle$ such that $o(a) = q + 1$, $o(b) = 2$. The group D contains $q + 1$ involutions which belong to different S_2-subgroups of G. Consequently, as there are exactly $q + 1$ S_2-subgroups in G, given a conjugate D^c ($\neq D$) of D in G, each involution in D commutes with exactly one involution from D^c. Thus, it is easy to deduce that if c is an involution in $G \backslash D$ which commutes with b, then for every i, $1 \leqslant i \leqslant q$, there exists a unique j ($=\gamma(i)$), $1 \leqslant j \leqslant q$, such that

$$a^i a^{\gamma(i)c} \text{ is an involution.}$$

Hence

$$SL(2, q) \text{ is a homomorphic image of}$$

$$\{a_1, a_2 \mid a_1^{q+1} = a_2^{q+1} = (a_1{}^i a_2^{\gamma(i)})^2 = 1, \text{ for } 1 \leqslant i \leqslant q\}.$$

We do not know of a proof which shows the above group to be a presentation for $SL(2, q)$. This questions was considered by Bussey in [2].

3. A Finiteness Criterion

Before proceeding to the main result of this section we need to introduce a combinatorial notion which seems to be novel and interesting in itself.

Definition 3.1. Let A and B be nonempty sets, H be a permutation group on A, and γ be some function from A into B. Then the quadruple $(H; A, B; \gamma)$ is called *special* provided

$$|\gamma(\gamma^{-1}(S)h)| \geqslant |S|,$$

for all nonempty subsets S of B, and all $h \in H$.

Actually, we will be dealing with a more particular form of this notion.

DEFINITION 3.2. Let H and K be finite groups, and let $\gamma: H \to K$ be some function. Then the triple $(H, K; \gamma)$ is said to be *special* provided γ satisfies

$$\text{(i)} \quad \gamma(1) = 1, \gamma^{-1}(1) = \{1\},$$
$$\text{(ii)} \quad |\gamma(\gamma^{-1}(S)h)| \geqslant |S|,$$

for all subsets S of K such that $1 \in S$, and for all $h \in H$.

Remarks 3.3. (i) Let $(H; A, B; \gamma)$ be special and suppose B is a finite set. Then γ is an onto function.

(ii) Let A and B be finite sets, and let H be a permutation group of A. Suppose $|A| = \kappa |B|$ for some integer κ. Then any partition of A into κ-subsets provides us with a map γ such that $(H; A, B; \gamma)$ is special.

(iii) Let H and K be finite groups such that $|H| = 1 + \kappa(|K| - 1)$ for some integer κ. Then any partition of $H\backslash\{1\}$ into κ-subsets leads to a map $\gamma: H \to K$ such that $(H, K; \gamma)$ is special. In particular, if $|H| = |K|$, then $(H, K; \gamma)$ is special for every bijection $\gamma: H \to K$ such that $\gamma(1) = 1$.

(iv) Let H and K be cyclic groups having orders 4 and 3, respectively. Then one can check directly that there is no map $\gamma: H \to K$ such that $(H, K; \gamma)$ is special.

DEFINITION 3.4. Let m, n be integers such that $1 < m \leqslant n$. Let $r = (m-1)(n-1)$ and define

$$\tau(s) = \begin{cases} m + n - 2 & \text{if } s \text{ is odd} \\ 2 & \text{if } s \text{ is even,} \end{cases}$$

for $1 \leqslant s \leqslant m$. Furthermore, define

$$f(m, n) = 1 + \sum_{s=1}^{m} \frac{r^{[s/2]}\tau(s)}{(s-1)!}.$$

We note that

$$f(m, n) \leqslant 1 + 2(n-1) \sum_{s=0}^{m-1} \frac{(n-1)^s}{s!} \leqslant (n-1)e^n,$$

where e is the base of the natural logarithms.

THEOREM 3.5. *Let $(H, K; \gamma)$ be a special triple, where H and K are finite groups of orders m and n, respectively, and such that $1 < m \leqslant n$. Let δ, ϵ be functions, where $\delta\colon H \to K$, $\epsilon\colon H \to H$. Then the group defined by*

$$\mathscr{X}(H, K; \gamma, \delta, \epsilon) = \{H, K \mid h\gamma(h)\, \epsilon(h)^{-1}\, \delta(h)^{-1} = 1, \forall h \in H\}$$

is finite of order at most $f(m, n)$.

Proof. Denote $\mathscr{X}(H, K; \gamma, \delta, \epsilon)$ by $\mathscr{X}$, and let ϕ be the canonical homomorphism from $H * K$ onto $\mathscr{X}$.

Let $\omega \in H * K$. Then, $\omega = h_1k_2h_3 \cdots k_{s-1}h_s$ for some $h_i \in H$ $(i = 1, 3,..., s)$, $k_j \in K$ $(j = 2,..., s - 1)$, and $s \geqslant 1$. Since the first and last elements in the expression for ω are from H or K, accordingly every nontrivial reduced word can be classified in a natural way as one of four types. Let $l(\omega)$ denote the syllable length of the reduced form of ω (see [9], p. 182).

Given ω_1, $\omega_2 \in H * K$, we say that ω_1 is equivalent to ω_2, and write $\omega_1 \equiv \omega_2$, provided $\phi(\omega_1\omega_2^{-1}) = 1$. Given $\omega \in H * K$, we say that ω is minimal provided

$$\omega' \in H * K, \qquad \omega' \equiv \omega \Rightarrow l(\omega') \geqslant l(\omega).$$

We claim that every minimal word ω of syllable length $s \geqslant 2$ is equivalent to $s - 1$ minimal words of the same length and type and such that all of these have distinct end elements. Naturally, this will imply that each word in $H * K$ is equivalent to another of length at most m, which in turn will imply that $\mathscr{X}$ is a finite group.

The proof proceeds by induction on $l(\omega)$. The case $l(\omega) = 2$ is trivial. Suppose $l(\omega) = 3$. Then,

$$\omega = k_1h_2k_3 \qquad \text{or} \qquad h_1k_2h_3$$

for some h_1, h_2, $h_3 \in H^{\#}$, and some k_1, k_2, $k_3 \in K^{\#}$. In the first case, we have

$$\begin{aligned}\omega = k_1h_2k_3 &= k_1(h_2\gamma(h_2))(\gamma(h_2)^{-1}\, k_3)\\ &\equiv k_1(\delta(h_2)\, \epsilon(h_2))(\gamma(h_2)^{-1}\, k_3)\\ &\equiv (k_1\delta(h_2))\, \epsilon(h_2)(\gamma(h_2)^{-1}\, k_3),\end{aligned}$$

and clearly, as $\gamma(h_2) \neq 1$, $k_3 \neq \gamma(h_2)^{-1}\, k_3$. In the second case, we have for any $h \in \gamma^{-1}(k_2^{-1})$,

$$\begin{aligned}\omega = h_1k_2h_3 &= h_1(k_2h^{-1})\, hh_3\\ &\equiv h_1(\epsilon(h)^{-1}\, \delta(h)^{-1})\, hh_3\\ &\equiv (h_1\epsilon(h)^{-1})\, \delta(h)^{-1}(hh_3),\end{aligned}$$

and clearly, as $1 \notin \gamma^{-1}(k_2^{-1})$, the set $\{h_3\} \cup \gamma^{-1}(k_2^{-1})\, h_3$ contains at least two distinct elements.

Let $l(\omega) = s > 3$, and suppose that the claim is true for minimal words ω' such that $l(\omega') < s$. There are two cases to be considered depending upon whether the end element of ω is from K or from H. We will assume that the first element of ω is from H in both cases.

Case 1. Suppose $\omega = h_1 k_2 \cdots h_{s-3} k_{s-2} h_{s-1} k_s$. Then, $\omega_0 = h_1 k_2 \cdots h_{s-3} k_{s-2}$ is also a minimal word. By induction, ω_0 is equivalent to the words

$$h_1^{(i)} k_2^{(i)} \cdots h_{s-3}^{(i)} k_{s-2}^{(i)}, \qquad 1 \leqslant i \leqslant s-3,$$

where the set

$$\{k_{s-2}^{(i)} \mid 1 \leqslant i \leqslant s-3\} \text{ has cardinality } s-3.$$

Next we define

$$\omega^{(i)} = h_1^{(i)} \cdots h_{s-3}^{(i)} k_{s-2}^{(i)} h_{s-1} k_s, \qquad 1 \leqslant i \leqslant s-3.$$

Since for each i, and for every $h \in \gamma^{-1}((k_{s-2}^{(i)})^{-1})$, $h_{s-3}^{(i)} k_{s-2}^{(i)} h_{s-1}$ is equivalent to $(h_{s-3}^{(i)} \epsilon(h)^{-1})\, \delta(h)^{-1}(hh_{s-1})$, we get that $\omega^{(i)} \equiv v^{(i)}(h)$, where $v^{(i)}(h)$ is defined as $h_1^{(i)} k_2^{(i)} \cdots k_{s-4}^{(i)}(h_{s-3}^{(i)} \epsilon(h)^{-1})\, \delta(h)^{-1}(hh_{s-1})\, k_s$. Furthermore, we define

$$W^{(i)} = \{v^{(i)}(h) \mid h \in \gamma^{-1}((k_{s-2}^{(i)})^{-1})\},$$

for $1 \leqslant i \leqslant s-3$, and let

$$W = \{\omega\} \cup W^{(1)} \cup \cdots \cup W^{(s-3)},$$
$$S = \{1, (k_{s-2}^{(1)})^{-1}, \ldots, (k_{s-2}^{(s-3)})^{-1}\}.$$

Then, the elements of H in the $(s-1)$th position of the elements of W form the set $\gamma^{-1}(S)\, h_{s-1}$. On substituting the last segments of length 3 of the elements of W,

$$k_{s-2} h_{s-1} k_s, \qquad \delta(h)^{-1}(hh_{s-1})\, k_s,$$

where $h \in \gamma^{-1}((k_{s-2}^{(i)})^{-1})$ and $1 \leqslant i \leqslant s-3$, by their respective equivalents

$$(k_{s-2}\delta(h_{s-1}))\, \epsilon(h_{s-1})(\gamma(h_{s-1})^{-1}\, k_s),$$
$$(\delta(h)^{-1}\, \delta(hh_{s-1}))\, \epsilon(hh_{s-1})(\gamma((hh_{s-1})^{-1})\, k_s),$$

a new set W' of words equivalent to ω is obtained. Their end elements form the set $E_0 = \gamma(\gamma^{-1}(S)\, h_{s-1})^{-1}\, k_s$. As

$$|\, \gamma(\gamma^{-1}(S)\, h_{s-1})| \geqslant |\, S \,| = s-2,$$

we have $|E_0| \geqslant s-2$. Since $k_s \notin E_0$, $W'' = \{\omega\} \cup W'$ contains at least $s-1$ words equivalent to ω, all having the same length and type, and whose end elements are all distinct.

Case 2. Suppose $\omega = h_1 k_2 \cdots k_{s-1} h_s$. As the proof of this case is similar to that of the first one, we will omit it. Indeed, this case is simpler since induction may be applied to $\omega_0 = h_1 k_2 \cdots k_{s-1}$.

Now we prove that any minimal word of length $s \geqslant 1$ is equivalent to at least $(s-1)!$ distinct minimal words of the same length and type.

The proof proceeds by induction on $l(\omega) = s$. The cases $s = 1, 2$ are trivial. Thus, suppose $s \geqslant 3$ and $\omega = h_1 k_2 \cdots k_s$. By the above argument, ω is equivalent to some minimal words

$$\omega^{(i)} = h_1^{(i)} k_2^{(i)} \cdots k_s^{(i)}, \qquad 0 \leqslant i \leqslant s-2,$$

such that

$$l(\omega) = l(\omega^{(i)}) \qquad \text{for every } i,$$

and $\{k_s^{(i)} \mid 0 \leqslant i \leqslant s-2\}$ has cardinality $s-1$.

Define $\omega_1^{(i)} = \omega^{(i)}(k_s^{(i)})^{-1}$ for $0 \leqslant i \leqslant s-2$. Then, $\omega_1^{(i)}$ is a minimal word and $l(\omega_1^{(i)}) = l(\omega) - 1$, for every i. By induction, $\omega_1^{(i)}$ is equivalent to distinct minimal words $\omega_j^{(i)}$, $1 \leqslant j \leqslant (s-2)!$, of the same length and type. Hence, the set

$$\{\omega_{(i,j)} \mid \omega_{(i,j)} = \omega_j^{(i)} k_s^{(i)}, 0 \leqslant i \leqslant s-2, 1 \leqslant j \leqslant (s-2)!\}$$

contains $(s-1)!$ distinct minimal words equivalent to ω.

We observe that the number of nontrivial words of length s in $H * K$ is

$$(n-1)^{[s/2]}(m-1)^{[s/2]}(m+n-2) \qquad \text{when } s \text{ is odd,}$$

and

$$2(n-1)^{s/2}(m-1)^{s/2} \qquad \text{when } s \text{ is even.}$$

It follows immediately that the number of nonequivalent reduced words is at most

$$1 + \sum_{\substack{s:\text{odd} \\ 1 \leqslant s \leqslant m}} \frac{(n-1)^{[s/2]}(m-1)^{[s/2]}(m+n-2)}{(s-1)!} + 2 \sum_{\substack{s:\text{even} \\ 1 \leqslant s \leqslant m}} \frac{(n-1)^{s/2}(m-1)^{s/2}}{(s-1)!}.$$

Hence, $|\mathscr{X}| \leqslant f(m, n)$.

Now we prove Theorem B in the form of

COROLLARY 3.6. *Assume $m = n \geqslant 2$ in the previous theorem. Then, $|\mathscr{X}| < ne^{n-1}$.*

Proof. We may assume $n > 2$; for if $n = 2$, then $\mathscr{X}$ is abelian and $|\mathscr{X}| \leqslant 4$.

Let n_i denote the number of minimal nonequivalent words in $H * K$ of length i. Clearly, $n_0 = 1$ and $n_1 \leqslant 2(n-1)$. Also,

$$n_2 \leqslant 2(n-1)^2 - (n-1) = (n-1)(2n-3),$$

since the set

$$\{hk \mid hk \equiv k'h' \text{ for some } k' \in K,\ h' \in H'\}$$

has cardinality at least $n - 1$. Let $k, k' \in K$, $h \in H$ and khk' a minimal word. Then,

$$l(khk') = 3 \Rightarrow k' \neq 1, \gamma(h).$$

Thus, $n_3 \leqslant \frac{1}{2}(n-1)(2n-3)(n-2)$. Likewise, we have $n_i \leqslant [1/(i-1)!]$ $(n-1)(2n-3)(n-2)^{(i-2)}$ for $i \geqslant 3$. It follows then that

$$|\mathscr{X}| \leqslant 1 + \sum_{i=1}^{n} n_i,$$

$$|\mathscr{X}| < 1 + 2(n-1) + (n-1)(2n-3)\frac{e^{n-2}-1}{n-2},$$

and

$$|\mathscr{X}| < ne^{n-1}.$$

Example 3.7. In view of the results obtained, it is natural to consider groups generated by a system of finite subgroups which are weakly permutable among themselves.

Let m be a non-negative integer and let

$$\begin{aligned}\tilde{G}(m) = \{\tau_i\ (0 \leqslant i \leqslant m) \mid \tau_i^2 = 1\ (0 \leqslant i \leqslant m),\\ (\tau_0\tau_i)^3 = 1\ (1 \leqslant i \leqslant m),\ (\tau_i\tau_j)^2 = 1\ (0 < i < j \leqslant m)\}.\end{aligned}$$

Then, $\tilde{G}(m)$ is a Coxeter group whose graph is

$$\begin{array}{ccc} & & \cdot \\ \cdot & & 2 \\ & & / \ 2 \\ \cdot & 0 & -\ 1 \\ & & \backslash \\ \cdot & & m \\ & & \cdot \end{array}$$

As is well-known (see [4], p. 141), $\tilde{G}(m)$ is finite only for $0 \leqslant m \leqslant 3$. Now let $a_i = \tau_0\tau_i$ $(1 \leqslant i \leqslant m)$, and let $G(m) = \langle a_1, a_2, \ldots, a_m\rangle$. Then, $G(m)$ is

τ_0-invariant, $[\tilde{G}(m) : G(m)] = 2$, and as is easily verifiable the following relations hold in $G(m)$,

$$a_i^3 = 1 \quad (1 \leqslant i \leqslant m), \qquad (a_i a_j^{-1})^2 = 1 \quad (1 \leqslant i < j \leqslant m).$$

Thus we conclude that $G(m)$ is generated by $\langle a_1 \rangle, \langle a_2 \rangle, \ldots, \langle a_m \rangle$ which are weakly permutable among themselves, and $G(m)$ is finite only for $1 \leqslant m \leqslant 3$.

4. $\mathscr{X}(H) = \{H, H^\psi \mid [h, h^\psi] = 1, \forall h \in H\}$

The purpose of this section is to study in detail the structure of $\mathscr{X}(H)$, where H and H^ψ are isomorphic groups through the isomorphism $\psi: H \to H^\psi$.

Toward this end, we will consider the following subgroups of $\mathscr{X}(H)$,

$$D(H) = [H, H^\psi], \qquad L(H) = [H, \psi],$$
$$W(H) = D(H) \cap L(H), \qquad L_1(H) = [L(H), H],$$
$$L_2(H) = [L(H), H^\psi], \qquad R(H) = [H, L(H), H^\psi].$$

We will also consider $\mathbf{M}(H) = W(H)/R(H)$, which will be shown to make sense, and to have $M(H)$ as a homomorphic image. At times, in order to avoid overloading the notation, we will omit the H from these symbols.

4.1. *Preliminary Results*

We begin with three lemmas the proofs of which are obvious and will be omitted.

LEMMA 4.1.1. *Let A and B be subgroups of a group G. Then,*

$$[A, B] \lhd \langle A, B \rangle.$$

Suppose that there exists an epimorphism λ: $A \to B$. Then,

$$[A, \lambda] \lhd \langle A, B \rangle, \qquad [A, \lambda]B = \langle A, B \rangle.$$

LEMMA 4.1.2. *The group $\mathscr{X}(H)$ admits an automorphism ψ' which interchanges h with h^ψ for all $h \in H$. The symbols ψ' and ψ will be identified.*

LEMMA 4.1.3. *The subgroups $D(H)$, $L(H)$, and $W(H)$ are normal and ψ-invariant in $\mathscr{X}(H)$.*

We recall that we introduced $\tilde{H}$ and $T(\tilde{H})$ in Section 2.3, where $\tilde{H}$ denoted a convering group of H.

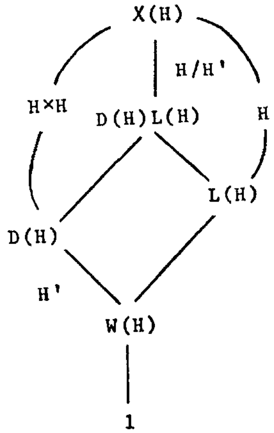
PROPOSITION 4.1.4. *There exists $\lambda_{\tilde{H}}$: $\mathscr{X}(H) \to T(\tilde{H})$, an epimorphism which satisfies*

$$\ker(\lambda_{\tilde{H}}) \subseteq W(H), \qquad \lambda_{\tilde{H}}(W(H)) = \overline{1 \times Z \times 1}.$$

Also, $\ker(\lambda_H) = W(H)$. *Furthermore,*

$$D(H) \cap H = L(H) \cap H = 1, \qquad \mathscr{X}(H) = L(H)H,$$

and the following diagram holds



Proof. Define $\lambda_{\tilde{H}} = H \cup H^{\psi} \to T(\tilde{H})$ by λ: $h \to h$, $h^{\psi} \to h^{\psi}$ for all $h \in H$. Then as $\lambda(h)$ commutes with $\lambda(h^{\psi})$ for all $h \in H$, λ is extendable to an epimorphism $\lambda_{\tilde{H}}$: $\mathscr{X}(H) \to T(\tilde{H})$ such that

$$\lambda_{\tilde{H}}(H) = H, \qquad \lambda_{\tilde{H}}(H^{\psi}) = H^{\psi},$$
$$\lambda_{\tilde{H}}(D(H)) = [H, H^{\psi}], \qquad \lambda_{\tilde{H}}(L(H)) = [H, \psi].$$

By remark (iv) in 2.3.1, there exist epimorphisms

$$\alpha\colon T(\tilde{H}) \to H \times H, \qquad \beta\colon T(\tilde{H}) \to H$$

with respective kernels $[H, H^{\psi}]$, and $[H, \psi]$. Since $\ker(\alpha\lambda_{\tilde{H}}) = D(H)$, $\ker(\beta\lambda_{\tilde{H}}) = L(H)$, it follows that

$$\ker(\lambda_{\tilde{H}}) \subseteq D(H) \cap L(H) = W(H).$$

Hence,

$$W(H)^{\lambda} = [H, H^{\psi}] \cap [H, \psi] = \overline{1 \times Z \times 1}.$$

Suppose that $\tilde{H} = H$; that is, $Z = 1$. Then clearly, $\ker(\lambda_H) = W(H)$.

The rest of the proposition follows from the structure of $T(H)$.

Parts (ii) and (iii) of Theorem C are established by the previous proposition together with Corollary 3.6

Remarks 4.1.5. (i) Better bounds for $|M(H)|$ when H is finite, will be obtained in 4.4.

(ii) In view of the previous proposition and its corollary, it is clear that a better description of $\mathscr{X}(H)$ depends crucially upon that of $W(H)$.

Concerning the action of H on $W(H)$, we will give an example in 4.2, where this action is not nilpotent.

LEMMA 4.1.6 (Commutator Identities). *Let* $h_1, h_2, h_3 \in H$. *Then,*

(i) $[h_1^\psi, h_2] = [h_1, h_2^\psi]$,

(ii) $[h_1, \psi]$ *commutes with* $[h_2, h_3^\psi]$,

(iii) $[h_1, h_2^\psi] = [\psi, h_2, h_1][h_1, h_2]$.

Proof. Consider the following calculations:

$$1 = [h_1h_2, (h_1h_2)^\psi] = [h_1, (h_1h_2)^\psi]^{h_2}[h_2, (h_1h_2)^\psi], \tag{1}$$

$$\begin{aligned}[h_1, (h_1h_2)^\psi] &= [h_1, h_1^\psi h_2^\psi] = [h_1, h_2^\psi][h_1, h_1^\psi]^{h_2^\psi} \\ &= [h_1, h_2^\psi],\end{aligned} \tag{2}$$

$$[h_2, (h_1h_2)^\psi] = [h_2, h_1^\psi]^{h_2^\psi}, \tag{3}$$

$$1 = [h_1, h_2^\psi]^{h_2}[h_2, h_1^\psi]^{h_2^\psi} \text{ (substitute (2), (3) in (1))}, \tag{4}$$

$$1 = [h_1, h_2^\psi]^{h_2^{-\psi}}[h_2, h_1^\psi]^{h_2^{-1}} \text{ (conjugate (4) by } h_2^{-1}h_2^{-\psi}), \tag{5}$$

$$1 = [h_2^{-\psi}, h_1][h_2^{-1}, h_1^\psi]^{-1}, \tag{6}$$

$$[h_1, h_2^{-\psi}] = [h_1^\psi, h_2^{-1}]. \tag{7}$$

Thus part (i) is proved.

Part (ii) is derived from

$$\begin{aligned}[h_1h_2, h_3^\psi] &= [h_1, h_3^\psi]^{h_2}[h_2, h_3^\psi], \\ &= [h_1, h_3^\psi]^{h_2}[h_2^\psi, h_3] \quad \text{(by (i))},\end{aligned} \tag{8}$$

$$\begin{aligned}[h_1h_2, h_3^\psi] &= [(h_1h_2)^\psi, h_3] \qquad \text{(by (i))}, \\ &= [h_1^\psi, h_3]^{h_2^\psi}[h_2^\psi, h_3] \\ &= [h_1, h_3^\psi]^{h_2^\psi}[h_2^\psi, h_3] \quad \text{(by (i))},\end{aligned} \tag{9}$$

$$[h_1, h_3^\psi]^{h_2} = [h_1, h_3^\psi]^{h_2^\psi} \text{ (by (8) and (9))}, \tag{10}$$

$$[h_1, h_3^\psi]^{h_2^{-1}h_2^\psi} = [h_1, h_3^\psi] \quad \text{(by (10))}. \tag{11}$$

Part (iii) follows from

$$\begin{aligned}[h_1, h_2^{\psi}] &= [h_1, h_2[h_2, \psi]] \\ &= [h_1, [h_2\psi]][h_1, h_2]^{[h_2,\psi]},\end{aligned} \tag{12}$$

$$\begin{aligned}[h_1, h_2^{\psi}] &= [h_1, [h_2, \psi]]^{[\psi,h_2]}[h_1, h_2] \\ &\quad \text{(by conjugating (12) by } [\psi, h_2]\text{, and using (ii)),}\end{aligned} \tag{13}$$

$$[h_1, h_2^{\psi}] = [\psi, h_2, h_1][h_1, h_2]. \tag{14}$$

PROPOSITION 4.1.7. *The groups $D(H)$, $L(H)$, and $W(H)$ satisfy*

(i) *ψ centralizes $D(H)$,*

(ii) *$D(H)$ centralizes $L(H)$,*

(iii) *$W(H)$ consists of all the elements of $\mathscr{X}(H)$ of the form $[h_1, h_2^{\psi}] \cdots [h_s, h_{s+1}^{\psi}]$, where $h_1, h_2, \ldots, h_{s+1} \in H$ and s is an odd natural number such that $[h_1, h_2] \cdots [h_s, h_{s+1}] = 1$.*

Proof. Parts (i) and (ii) follow directly from the corresponding parts of the previous lemma. Let $\omega \in D(H)$. Then,

$$\omega = [h_1, h_2^{\psi}] \cdots [h_s, h_{s+1}^{\psi}],$$

for some $h_1, h_2, \ldots, h_{s+1} \in H$, and s an odd natural number. By part (iii) of the previous lemma,

$$\begin{aligned}\omega &= [\psi, h_2, h_1][h_1, h_2] \cdots [\psi, h_{s+1}, h_s][h_s, h_{s+1}] \\ &= c[h_1, h_2] \cdots [h_s, h_{s+1}],\end{aligned}$$

where $c \in L(H)$. Since $H \cap L(H) = 1$, we have

$$\omega \in W(H) \Leftrightarrow [h_1, h_2] \cdots [h_s, h_{s+1}] = 1.$$

COROLLARY 4.1.8. *Let H be a solvable group having finite derived length k. Then, $\mathscr{X}(H)$ is a solvable group of derived length at most $k + 1$.*

Proof. We have by Proposition 4.1.4, that

$$\mathscr{X}(H)^{(k)} \subseteq D(H), L(H).$$

Now, as $W(H) = D(H) \cap L(H)$ is abelian, it follows that

$$\mathscr{X}(H)^{(k+1)} = [\mathscr{X}(H)^{(k)}, \mathscr{X}(H)^{(k)}] = 1.$$

The above corollary proves one of the inheritance properties in Theorem C.

LEMMA 4.1.9. *Suppose x and y are commuting elements of H. Then,*

(i) $[x, y^{\psi}] \in W(H)$,

(ii) $[x^i, y^{\psi}] = [x, y^{\psi}]^i$, *for all* $i \in Z$.

Furthermore, if x and y have finite relatively prime orders then $[x, y^{\psi}] = 1$ holds.

Proof. Part (i) is but a special case of part (iii) of the previous proposition. Part (ii) follows from

$$\begin{aligned}[x^2, y^{\psi}] &= [x, y^{\psi}]^x[x, y^{\psi}] \\ &= [x, y^{\psi}]^{x^{\psi}}[x, y^{\psi}] = [x, y^{\psi}]^2,\end{aligned}$$

and

$$\begin{aligned}1 = [xx^{-1}, y^{\psi}] &= [x, y^{\psi}]^{x^{-1}}[x^{-1}, y^{\psi}] \\ &= [x, y^{\psi}][x^{-1}, y^{\psi}].\end{aligned}$$

Suppose x and y have finite orders. Then, since $[x, y^{\psi}] = [x^{\psi}, y]$, we have

$$o([x^{\psi}, y]) \mid (o(x), o(y)).$$

LEMMA 4.1.10. *Let $L = L(H)$, $L_1 = L_1(H) = [L, H]$, and let $L_2 = L_2(H) = [L, H^{\psi}]$. Then,*

(i) $L_1, L_2 \lhd \mathscr{X}(H)$,

(ii) $L_1 = L \cap (DH')$, $L_2 = L \cap (DH'^{\psi})$,

(iii) $L_1 \cap L_2 = W$,

(iv) $D \subseteq L_1H'$, *and the projections of D into L_1 and H' are both onto,*

(v) $L_1 \cong D \cong L_2 \cong H'$ *in $\mathscr{X}(H)$ modulo W.*

Proof. (i), (ii). Let $h_1, h_2 \in H$. By part (iii) of Lemma 4.16,

$$[\psi, h_2, h_1] = [h_1, h_2^{\psi}][h_1, h_2]^{-1}.$$

Thus

$$[\psi, h_2, h_1] \in DH'.$$

Now, since $DH' \lhd \mathscr{X}(H)$, we have

$$L_1 = [L, H] \subseteq (DH') \cap L.$$

To complete the proof of (ii), consider $x \in (DH') \cap L$. Then, $x = dh'$ for some $d \in D$ and $h' \in H'$. However, $d = l_0h_0'$ for some $l_0 \in [L, H]$ and $h_0' \in H'$. Hence, $x = l_0(h_0'h') = l_0$, since $x \in L$.

Part (i) follows from part (ii) and from the fact that DH' and DH'^{ψ} are normal subgroups of $\mathscr{X}$ (considered $\alpha\lambda_H$ of 4.1.4).

(iii) We have

$$L_1 \cap L_2 = L \cap (DH') \cap (DH'^{\psi}),$$

and

$$W \subseteq L_1 \cap L_2 .$$

Now, by the fact that $HH^{\psi} \cap D = 1$, $L_1 \cap L_2 = W$ follows.

(iv) The fact that $D \subseteq L_1H'$ and that D projects onto H' are clear. The assertion that D projects onto L_1 follows from part (ii) of this lemma.

(v) The epimorphism $\lambda_H: \mathscr{X}(H) \to T(H)$ of Proposition 4.1.4, maps L_1 onto $H' \times 1 \times 1$, D onto $1 \times H' \times 1$ and L_2 onto $1 \times 1 \times H'$. Since $\ker(\lambda_H) = W$, the proof of this part follows.

LEMMA 4.1.11. *Let $R = R(H) = [H, L(H), H^{\psi}]$. Then,*

(i) *$R \lhd \mathscr{X}(H)$, R is ψ-invariant,*

(ii) *for any covering group $\tilde{H}$ of H, $[W, H] \subseteq R \subseteq \ker(\lambda_{\tilde{H}}) \subseteq W$,*

(iii) *$[h_1, h_2^{\psi}]^{h_3} = [h_1^{h_3}, h_2^{h_3\psi}]$ holds in $\mathscr{X}$ modulo R, for any $h_1, h_2, h_3 \in H$,*

(iv) *$[H', Z(H)^{\psi}] \subseteq R$.*

Proof. (i) We have: $L_1 = [H, L(H)]$, $L_1 \lhd \mathscr{X}(H)$, $R = [L_1, H^{\psi}]$, $R^H \subseteq [L_1, DH^{\psi}] \subseteq [L_1, H^{\psi}] = R$. Hence, as H^{ψ} also normalizes R, $R \lhd \mathscr{X}(H)$. Since $[H, H^{\psi}, L] = 1$, it follows that

$$[H, L, H^{\psi}] = [H^{\psi}, L, H],$$

and consequently, R is ψ-invariant.

(ii) Since $W \subseteq L_1$, we have that

$$[W, H] \subseteq [L_1, H] = R.$$

We consider $\tilde{H}$ any covering group of H, $\tilde{T} = \tilde{H} \times \tilde{H} \times \tilde{H}$, and $u_1, u_2, v_1, v_2, w_1, w_2 \in \tilde{H}$. Thus, $[(u_1, u_2, 1), (v_1, 1, v_2), (1, w_1, w_2)] = 1$. Consequently, $[H, L, H^{\psi}] \subseteq \ker(\lambda_{\tilde{H}})$.

(iii) Let $h_1, h_2, h_3 \in H$. Then,

$$[h_1, h_2^{\psi}]^{h_3} = [h_1^{h_3}, h_2^{\psi h_3}], \tag{1}$$

$$h_2^{\psi h_3} = (h_2^{h_3\psi})^l, \qquad \text{where} \quad l = h_3^{-\psi}h_3, \tag{2}$$

$$\begin{aligned} [h_1^{h_3}, h_2^{\psi h_3}] &= [h_1^{h_3}, h_2^{h_3\psi l}] \quad ((2) \text{ in } (1)), \\ &= [h_1^{h_3}, h_2^{h_3\psi}l_0], \qquad \text{where} \quad l_0 = [h_2^{h_3\psi}, l], \end{aligned} \tag{3}$$

$$[h_1^{h_3}, h_2^{\psi h_3}] = [h_1^{h_3}, l_0][h_1^{h_3}, h_2^{h_3\psi}]^{l_0}. \tag{4}$$

As $l_0 \in [H^\psi, L]$ and $[h_1^{h_3}, h_2^{h_3\psi}] \in D$, we have from (4) that

$$[h_1^{h_3}, h_2^{\psi h_3}] \in R[h_1^{h_3}, h_2^{h_3\psi}].$$

(iv) Let $z \in Z(H)$, x, $h \in H$. By Lemma 4.1.9, $[x, z^\psi] \in W$. Also, the following equalities hold modulo R,

$$\begin{aligned}[x, z^\psi]^h &= [x^h, z^{h\psi}] \qquad \text{(by (iii))},\\ &= [x[x, h], z^\psi] \qquad\qquad (1)\\ &= [x, z^\psi]^{[x,h]}[x, h, z^\psi],\\ [x, z^\psi] &= [x, z^\psi][x, h, z^\psi] \qquad \text{(since } [W, H] \subseteq R), \qquad (2)\\ [x, h, z^\psi] &= 1. \qquad\qquad (3)\end{aligned}$$

Hence, $[H', Z(H)^\psi] \subseteq R$.

In the following three propositions, we view $\mathscr{X}\colon H \to \mathscr{X}(H)$ as an operation on the class of groups, and study its behaviour with respect to taking subgroups, homomorphic images, and direct product of groups. Actually, in the case of taking subgroups K of H, we study $\langle K, K^\psi\rangle$ in $\mathscr{X}(H)$, which is evidently a homomorphic image of $\mathscr{X}(K)$, yet not necessarily isomorphic to it.

PROPOSITION 4.1.12. *Suppose $K \leqslant H_1 \leqslant H$ is a chain of subgroups of $\mathscr{X}(H)$. Then,*

(i) $[K, H_1^\psi] \lhd \langle H_1, H_1^\psi\rangle$,

(ii) $\langle K, K^\psi\rangle \cap D(H) = [K, H^\psi]$,

(iii) $\langle K, K^\psi\rangle \cap L(H) = [K, \psi]$.

Furthermore, suppose that $K \lhd H$. Then,

(iv) (*the normal closure*) $\langle K, K^\psi\rangle^{\mathscr{X}(H)} = \langle K, K^\psi\rangle^H = [K^\psi, H]\langle K, K^\psi\rangle$.

Proof. (i) By Lemma 4.1.1, $[K^\psi, H_1] \lhd \langle K^\psi, H_1\rangle$. Since ψ centralizes $D(H)$, it follows that $[K^\psi, H_1] = [K, H_1^\psi]$ and therefore, $[K^\psi, H_1] \lhd \langle H_1, H_1^\psi\rangle$.

(ii) Denote $\langle K, K^\psi\rangle$ by $\hat{K}$. Clearly, $[K, K^\psi] \subseteq \hat{K} \cap D(H)$. We recall the epimorphism $\alpha\lambda_H\colon \mathscr{X}(H) \to H \times H$ determined by $h \to (h, 1)$, $h^\psi \to (1, h)$, for all $h \in H$. Since $\ker(\alpha\lambda_H) = D(H)$, $\alpha\lambda_H$ induces an epimorphism from $\hat{K}/[\hat{K} \cap D(H)]$ into $K \times K$. Thus, we have the epimorphism sequence

$$\frac{\hat{K}}{\hat{K} \cap D(H)} \to K \times K \to \frac{\hat{K}}{[K, K^\psi]},$$

where

$$\begin{aligned}(\hat{K} \cap D(H))k &\to (k, 1) \to [K, K^\psi]k,\\ (\hat{K} \cap D(H))\, k^\psi &\to (1, k) \to [K, K^\psi]\, k^\psi,\end{aligned}$$

for all $k \in K$. The composition of the two epimorphisms has the effect

$$(\hat{K} \cap D(H))x \to [K, K^{\psi}]x,$$

for all $x \in K$. Hence

$$\hat{K} \cap D(H) \subseteq [K, K^{\psi}],$$

and the desired conclusion is reached.

(iii) The proof of this part is similar to that of (ii). Here, we have to consider $\beta\lambda_H$: $\mathscr{X}(H) \to H$, and the sequence

$$\frac{\hat{K}}{\hat{K} \cap L(H)} \to K \to \frac{\hat{K}}{[\psi, K]}.$$

(iv) The conclusion is reached through the following identities,

$$\begin{aligned} \langle K, K^{\psi}\rangle^{H} &= \langle K, (K^{\psi})^{H}\rangle = \langle K, [K^{\psi}, H], K^{\psi}\rangle, \\ &= \langle K, K^{\psi}\rangle[K^{\psi}, H] = \langle K, K^{\psi}\rangle[K, H^{\psi}], \\ &= \langle K, K^{\psi}\rangle^{H^{\psi}}. \end{aligned}$$

PROPOSITION 4.1.13. *Let H be a group and K a homomorphic image of H by an epimorphism ϕ: $H \to K$. Also, let $N = \ker(\phi)$. Then there exists a natural extension of ϕ to an epimorphism $\hat{\phi}$: $\mathscr{X}(H) \to \mathscr{X}(K)$ such that*

$$\text{(i)}\quad \hat{\phi}(D(H)) = D(K), \qquad \hat{\phi}(L(H)) = L(K),$$

$$\text{(ii)}\quad \ker(\hat{\phi}) = \langle N, N^{\psi}\rangle[N, H^{\psi}].$$

On letting $\hat{\phi}_D$ be the restriction of $\hat{\phi}$ to $D(H)$, we have

$$\text{(iii)}\quad \ker(\hat{\phi}_D) = [N, H^{\psi}],$$

$$\text{(iv)}\quad \hat{\phi}_D^{-1}(W(K)) = D(H) \cap (L(H)N),$$

$$\text{(v)}\quad \frac{D(H)}{\hat{\phi}_D^{-1}(W(K))} \cong \frac{D(K)}{W(K)} \cong \frac{H'}{H' \cap N},$$

and

$$\frac{\hat{\phi}_D^{-1}(W(K))}{W(H)} \cong H' \cap N.$$

Proof. The extension $\hat{\phi}$ of ϕ is determined by $\hat{\phi}(h) = \phi(h)$, and $\hat{\phi}(h^{\psi}) = \phi(h)^{\psi}$, for all $h \in H$. Thus, part (i) follows easily.

(ii) Evidently, $\langle N, N^\psi\rangle \subseteq \ker(\hat{\phi})$. We will denote the normal closure of $\langle N, N^\psi\rangle$ in $\mathscr{X}(H)$ by $\mathbf{N}$. Then, by part (iv) of the previous proposition,

$$\mathbf{N} = [N^\psi, H]\langle N, N^\psi\rangle.$$

Hence,

$$\mathscr{X}(H)/\mathbf{N} = \langle H\mathbf{N}/\mathbf{N}, H^\psi\mathbf{N}/\mathbf{N}\rangle.$$

On considering the composition of the epimorphisms

$$\mathscr{X}(K) \to \mathscr{X}(H)/\mathbf{N} \to \mathscr{X}(K),$$

where for all $k \in K$,

$$k \to \mathbf{N}\hat{\phi}^{-1}(k) \to k,$$
$$k^\psi \to \mathbf{N}\hat{\phi}^{-1}(k^\psi) \to k^\psi,$$

we realize that it is the identity transformation, and so, $\mathbf{N} = \ker(\hat{\phi})$ follows.

(iii) We have that

$$D(H) \cap \ker(\hat{\phi}) \supseteq [N, H^\psi],$$

from which we conclude,

$$D(H) \cap \ker(\hat{\phi}) = (D(H) \cap \langle N, N^\psi\rangle)[N, H^\psi].$$

Thus, by part (ii) of the previous proposition,

$$D(H) \cap \ker(\hat{\phi}) = [N, N^\psi][N, H^\psi] = [N, H^\psi].$$

(iv) We have that

$$[D(H)\ker(\hat{\phi}) \cap L(H)\ker(\hat{\phi})]^{\hat{\phi}} = D(K) \cap L(K) = W(K).$$

Let $u \in W(K)$. Then, $u = \hat{\phi}(v)$ for some

$$v \in D(H)\ker(\hat{\phi}) \cap L(H)\ker(\hat{\phi}).$$

Let $d \in D$, $l \in L$, and $\omega_1, \omega_2 \in \ker(\hat{\phi})$ such that $v = d\omega_1 = \omega_2 l$. Then,

$$u = \hat{\phi}(d) = \hat{\phi}(l).$$

We also have

$$\omega_1 = x_1 y_1, \qquad \omega_2 = x_2 y_2,$$

where

$$x_1, x_2 \in [N, H^\psi], \qquad \text{and} \qquad y_1, y_2 \in \langle N, N^\psi\rangle.$$

Therefore,

$$v = d\omega_1 = (dx_1)\,y_1 = x_2 y_2 l.$$

Now, on letting $v' = x_2^{-1}\,dx_1$, we get

$$v' = x_2^{-1}\,dx_1 = y_2 l y_1^{-1},$$

thus, we have

$$v' \in D(H) \cap (L(H)\langle N, N^{\psi}\rangle),$$

and

$$u = \hat{\phi}(v') = \hat{\phi}(d).$$

Since $L(H)\langle N, N^{\psi}\rangle = L(H)N$, we conclude that

$$v' \in D(H) \cap (L(H)N),$$

and consequently,

$$W(K) = (D(H) \cap (L(H)N))^{\hat{\phi}}$$

follows. Considering that

$$D(H) \supseteq [N, H^{\psi}], \qquad \text{and} \qquad N \lhd H,$$

we conclude,

$$L(H)N \supseteq [N, H^{\psi}].$$

Hence, it follows that

$$D(H) \cap L(H)N \supseteq \ker(\hat{\phi}_D),$$

and

$$\hat{\phi}_D^{-1}(W(K)) = D(H) \cap (L(H)N).$$

(v) Let π be the projection of $D(H)$ onto H', obtained from $D(H) \subseteq L(H)\,H'$ (see Lemma 4.1.10, part (iv)). Then we have, $\ker(\pi) = W(H)$, $\pi(D \cap LN) = H' \cap N$, and

$$\pi^{-1}(H' \cap N) = D \cap LN.$$

Let $\bar{\pi}$: $D(H) \to H'/(H' \cap N)$ be the map defined by

$$\bar{\pi}(d) = (H' \cap N)\,\pi(d),$$

for all $d \in D(H)$. Then,

$$\ker(\bar{\pi}) = D \cap LN,$$

which implies

$$\frac{D}{D \cap LN} \cong \frac{H'}{H' \cap N}.$$

Whenever the lowest term of the descending central series of a group G exists, it will be denoted by G_ν.

PROPOSITION 4.1.14. *Let G be a group. Suppose H and K are normal subgroups of G such that $G = H \oplus K$. Then,*

$$\text{(i)}\quad \mathscr{X}(G) = [H, K^\psi]\langle H, H^\psi\rangle\langle K, K^\psi\rangle,$$

$$\text{(ii)}\quad \langle H, H^\psi\rangle \cong \mathscr{X}(H), \qquad \langle K, K^\psi\rangle \cong \mathscr{X}(K),$$

$$\text{(iii)}\quad W(G) = [H, K^\psi] \oplus W(H) \oplus W(K).$$

Furthermore, on supposing G to be finite,

$$\text{(iv)}\quad [H, K^\psi] \cong \left[\frac{H}{H_\nu}, \left(\frac{K}{K_\nu}\right)^\psi\right],$$

where the latter commutator is taken inside $\mathscr{X}(H/H_\nu \times K/K_\nu)$.

Proof. Since $[H, K] = 1$, we get by Lemma 4.1.9, that $[H, K^\psi] \subseteq W(G)$. Also,

$$[H, K^\psi] \lhd \mathscr{X}(G)$$

holds, since $[H, K^\psi]$ is normalized by H, K^ψ, and by H^ψ, K.

Let $\rho\colon \mathscr{X}(G) \to \mathscr{X}(H) \times \mathscr{X}(K)$ be the epimorphism determined by $\rho(h) = (h, 1)$, $\rho(h^\psi) = (h^\psi, 1)$, $\rho(k) = (1, k)$, $\rho(k^\psi) = (1, k^\psi)$ for all $h \in H$, $k \in K$. Then,

$$\ker(\rho) = [H, K^\psi], \qquad \rho\langle H, H^\psi\rangle = \mathscr{X}(H),$$

and $\rho\langle K, K^\psi\rangle = \mathscr{X}(K)$. Also, since $\mathscr{X}(H)$ maps naturally onto $\langle H, H^\psi\rangle$, $(\ker \rho) \cap \langle H, H^\psi\rangle = 1$; therefore, we have that $\mathscr{X}(H) \cong \langle H, H^\psi\rangle$. Likewise, $\mathscr{X}(K) \cong \langle K, K^\psi\rangle$. Hence,

$$\mathscr{X}(G) = [H, K^\psi]\langle H, H^\psi\rangle\langle K, K^\psi\rangle,$$

$$W(G) \cong [H, K^\psi] \times W(H) \times W(K).$$

Suppose G is a finite group. Then, by Lemma 4.1.9, given $h \in H$, $k \in K$, we have $[h, k^\psi] = 1$, if $(o(h), o(k)) = 1$. Let $P \in \mathrm{Syl}_p(H)$. Then, P centralizes all the p'-elements of K^ψ. Thus, by varying the prime p, it follows that

$$H \text{ centralizes } K_\nu{}^\psi, \qquad K \text{ centralizes } H_\nu{}^\psi.$$

Hence we have, by part (iv) of Proposition 4.1.12,

$$\begin{aligned}\langle H_\nu, H_\nu^\psi\rangle^{\mathscr{X}(G)} &= [H_\nu, G^\psi]\langle H_\nu, H_\nu^\psi\rangle \\ &= [H_\nu, H^\psi]\langle H_\nu, H_\nu^\psi\rangle\end{aligned}$$

and is a subgroup of $\langle H, H^\psi\rangle$. Likewise,

$$\langle K_\nu, K_\nu^\psi\rangle^{\mathscr{X}(G)} = [K_\nu, K^\psi]\langle K_\nu, K_\nu^\psi\rangle,$$

and is a subgroup of $\langle K, K^\psi\rangle$.

The natural maps

$$\mu_1: H \to \frac{H}{H_\nu}, \qquad \mu_2: K \to \frac{K}{K_\nu}$$

induce an epimorphism

$$\mu: \mathscr{X}(G) \to \mathscr{X}\left(\frac{H}{H_\nu} \times \frac{K}{K_\nu}\right),$$

where

$$\ker \mu = \langle G_\nu, G_\nu^\psi\rangle^{\mathscr{X}(G)}.$$

Clearly, $[H_\nu, H^\psi]$ and $[K_\nu, K^\psi]$ are normal subgroups of $\mathscr{X}(H)$, and so

$$\begin{aligned}[G_\nu, G^\psi] &= [H_\nu, H^\psi][K_\nu, K^\psi] \\ &= D(G) \cap \ker \mu.\end{aligned}$$

Since $[H, K^\psi] = \ker \rho$, it follows that

$$\begin{aligned}[H, K^\psi] \cap \ker \mu &= [H, K^\psi] \cap D(G) \cap \ker \mu \\ &= \ker \rho \cap [G_\nu, G^\psi] = 1.\end{aligned}$$

Hence it follows that

$$[H, K^\psi] \cong \left[\frac{H}{H_\nu}, \left(\frac{K}{K_\nu}\right)^\psi\right],$$

where we have regarded H/H_ν, K/K_ν as subgroups of $H/H_\nu \times K/K_\nu$.

COROLLARY 4.1.15. *Let G be a finite nilpotent group and let $\{P_1, P_2, \ldots, P_k\}$ be the set of distinct Sylow p-subgroups of G. Then,*

$$\mathscr{X}(G) \cong \mathscr{X}(P_1) \times \mathscr{X}(P_2) \times \cdots \times \mathscr{X}(P_k).$$

Proof. We observe that in the proof of the previous proposition, the assumption $(|H|, |K|) = 1$ implies $[H, K^\psi] = 1$, and

$$\mathscr{X}(G) = \langle H, H_k\rangle \oplus \langle K, K^\psi\rangle.$$

Now the desired conclusion follows directly.

4.2. *H Abelian*

It is our purpose in this subsection to investigate the structure of $\mathscr{X}(H)$ for an abelian group H. Since $\mathscr{X}(H) = L(H)H$, the study will be directed toward a description of $L(H)$ and the action of H on it.

The structure of $\mathscr{X}(H)$ for finite abelian p-groups H, where p is a prime number, reveals basic differences between the cases p odd and $p = 2$. For instance, when H is an elementary abelian p-group, and $|H| = p^k$, $k \geqslant 1$, then

$$|\mathscr{X}(H)| = \begin{cases} p^{\frac{k(k-1)}{2}} p^{2k} & \text{for } p \text{ odd} \\ 2^{2^k-1}\, 2^k & \text{for } p = 2. \end{cases}$$

The case where p is odd will follow directly from Theorem 4.2.4. As for the case $p = 2$, here we have $(hh^\psi)^2 = 1$ for all $h \in H$ and easily then, $\mathscr{X}(H) \cong \mathscr{C}(H; 2)$ which has order $2^{|H|-1}\,|H|$.

THEOREM 4.2.1. *Let H be an abelian group. Then,*

(i) $D = W = [L, H]$, $R = [D, H] = [L, 2H]$,

(ii) *L is nilpotent of class* $\leqslant 2$, $L' \subseteq D \subseteq Z(L)$, $L' \subseteq Z(\mathscr{X})$,

(iii) $L' = D^2 = [H^2, H^\psi]$, $R^2 = 1$.

Proof. By Lemma 4.1.10, $D \subseteq [H, L]\, H'$ and the projection of D on $[H, L]$ is onto; thus,

$$D = [H, L] \subseteq L, \qquad D = L \cap D = W \subseteq Z(L).$$

By definition, $R = [H, L, H^\psi]$. Hence

$$R = [D, H^\psi] = [D, H] = [L, H, H] = [L, 2H].$$

As H centralizes L modulo $[L, H]$ $(=D)$, H^ψ also centralizes L modulo D. Hence,

$$L' \subseteq [\mathscr{X}, L] \subseteq D$$

follows. Since

$$[D, L] = [H, L, L] = 1 = [L, H, L] \text{ hold,}$$

we obtain

$$[L, L, H] = 1 \qquad \text{and} \qquad L' \subseteq Z(\mathscr{X}).$$

Therefore we have proved parts (i) and (ii). Part (iii) will be proved as a corollary of the following lemma. Given $h, h_1 \in H$, we define $v(h) = [h, \psi]$, and $v'(h, h_1) = [h_1, h^{\psi}]$; these being elements of $\mathscr{X}(H)$.

LEMMA 4.2.2. *Let H be an abelian group and let $h, h_1 \in H$, $i \in \mathbb{Z}$. Then,*

$$\begin{aligned} &\text{(i)} \quad v(h)\, v(h_1) = v(hh_1)\, v'(h, h_1), \\ &\text{(ii)} \quad h_1^{-1} v(h) h_1 = v(h)\, v'(h, h_1)^{-1}, \\ &\text{(iii)} \quad v'(h, h_1)^{i(i-1)} = [v(h), v(h_1)]^{i(i-1)/2}. \end{aligned}$$

Proof. We have

$$[h_1, h^{\psi}] = [h_1, h^{-1}h^{\psi}] = [h_1, [h, \psi]], \tag{1}$$

$$[hh_1, \psi] = [h, \psi]^{h_1}[h_1, \psi], \tag{2}$$

$$\begin{aligned} [hh_1, \psi] &= [h, \psi][h, \psi, h_1][h_1, \psi] \\ &= [h, \psi][h_1, \psi][h, \psi, h_1], \end{aligned} \tag{3}$$

for $[h, \psi, h_1] \in D \subseteq Z(L)$. Thus,

$$[hh_1, \psi] = [h, \psi][h_1, \psi][h_1, h^{\psi}]^{-1} \qquad \text{(by (3) and (1))}, \tag{4}$$

$$[h, \psi][h_1, \psi] = [hh_1, \psi][h_1, h^{\psi}], \qquad \text{which is part (i).} \tag{5}$$

We also have

$$h_1^{-1}[h, \psi]h_1 = [hh_1, \psi][h_1, \psi]^{-1} \qquad \text{(from (2))}, \tag{6}$$

$$h_1^{-1}[h, \psi]h_1 = [h, \psi][h_1, h^{\psi}]^{-1} \qquad \text{(by (6) and (4))}, \tag{7}$$

which is part (ii). Let $i \in \mathbb{Z}$. Then,

$$v(h^i) = [h^i, \psi] = h^{-i}h^{i\psi} = (h^{-1}h^{\psi})^i = v(h)^i. \tag{8}$$

Since $D \subseteq Z(L)$, we have

$$h^{-1}[h_1^{\psi}, h]h = h^{-\psi}[h_1^{\psi}, h]\, h^{\psi} = [h_1^{\psi}, h],$$

from which we may easily deduce

$$v'(h^i, h_1^j) = v'(h, h_1)^{ij} \tag{9}$$

for all $i, j \in \mathbb{Z}$. Clearly, we also have

$$v'(h, h_1) = v'(h_1, h)^{-1}. \tag{10}$$

Since L is nilpotent of class at most 2, by (8),

$$\begin{aligned}(v(h)\,v(h_1))^i &= v(h^i)\,v(h_1{}^i)[v(h_1), v(h)]^{i(i-1)/2},\\ &= v(h^ih_1{}^i)\,v'(h^i, h_1{}^i)[v(h_1), v(h)]^{i(i-1)/2}\\ &\quad \text{(by (i) of this lemma)}, \qquad (11)\\ &= v((hh_1)^i)\,v'(h, h_1)^{i^2}[v(h_1), v(h)]^{i(i-1)/2}\\ &\quad \text{(by (9))}.\end{aligned}$$

On the other hand, the following equalities hold

$$\begin{aligned}(v(h)\,v(h_1))^i &= (v(hh_1)\,v'(h, h_1))^i\\ &\quad \text{(by (i) of this lemma)},\\ &= v(hh_1)^i\,v'(h, h_1)^i,\\ &= v((hh_1)^i)\,v'(h, h_1)^i. \qquad (12)\end{aligned}$$

From (11) and (12) we produce

$$v'(h, h_1)^{i(i-1)} = [v(h), v(h_1)]^{i(i-1)/2},$$

which is part (iii) of the lemma.

Corollary 4.2.3. $L' = D^2 = [H^2, H^\psi]$, $R^2 = 1$.

Proof. Set $i = 2$ in part (iii) of the previous lemma. Then,

$$v'(h, h_1)^2 = [v(h), v(h_1)].$$

As D is abelian and is generated by $\{v'(h, h_1) \mid h, h_1 \in H\}$, we have

$$D^2 \subseteq L' \qquad \text{and} \qquad D^2 = L'.$$

By (9) of the same lemma,

$$\begin{aligned}v'(h, h_1)^2 = v'(h, h_1{}^2); \qquad &\text{therefore},\\ D^2 = [H^2, H^\psi].&\end{aligned}$$

Let $d \in D$, $h \in H$. Since $D^2\ (=L') \subseteq Z(\mathscr{X})$, we have

$$\begin{aligned}1 = [d^2, h] &= [d, h]^d[d, h]\\ &= [d, h]^2 \qquad \text{(for } D \text{ is abelian)}.\end{aligned}$$

Since $R = [D, H]$, we have obtained $R^2 = 1$.

THEOREM 4.2.4. *Let H be a finite abelian p-group (p a prime) of type $(a_1, a_2, \ldots, a_s)$ with $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_s$, and $s > 1$. Furthermore, let $u = a_2 + 2a_3 + \cdots + (s-1)a_s$, $u' = u - s(s-1)/2$. Then,*

(i) $\mathbf{M}(H) \cong M(H)$, $|M(H)| = p^u$,

$W(H)$ and $M(H)$ have exponents equal to p^{a_2},

(ii) *for odd p, $\mathscr{X}(H) \cong T(\tilde{H})$ for some full covering group $\tilde{H}$ of H,*

(iii) *for $p = 2$,*

$$|\mathscr{X}(H)| \text{ divides } |\mathscr{X}(H/H^2)|\,|\mathscr{X}(H^2)|\,\frac{|D(H)^2|}{|D(H)^4|},$$

where $D(H)^2$ has order $p^{u'}$ and is abelian of type $(a_2 - 1, 2 \times (a_3 - 1), \ldots, (s-1) \times (a_s - 1))$, and where $\mathscr{X}(H/H^2) \cong \mathscr{C}(H/H^2; 2)$.

Proof. Since L is a nilpotent group generated by $\{v(h) \mid h \in H\}$ and since from (8) of Lemma 4.2.2 we may conclude that $o(v(h)) \mid o(h)$ for all $h \in H$, it follows that L is a p-group. Hence, $\mathscr{X}$ is a p-group too.

(i) Since $R = [D, H]$, we have

$$D = \langle v'(h_i, h_j) \mid 1 \leqslant i < j \leqslant s \rangle + R.$$

By (9) of Lemma 4.2.2, $o(v'(h_1, h_j)) \mid p^{a_j}$ for $i < j$, and so $|D|$ divides $p^u |R|$.

Now we could either appeal to the well-known fact that the Schur Multiplier $M(H)$ has order p^u and is of type $(a_2, 2 \times a_3, \ldots)$, or to the fact that the construction of a covering group $\tilde{H}$ of H of order $|H|\,p^u$ is not too difficult to carry out. Since by Lemma 4.1.11, $M(H)$ is a factor group of $\mathbf{M}(H)$, we conclude that

$$M(H) \cong \mathbf{M}(H), \qquad \text{and} \qquad \exp(D) = p^{a_2}.$$

(ii) Let p be odd. Then, $R = 1$ and

$$|\mathscr{X}(H)| \text{ divides } p^u\,|H|^2 \text{ which is } |T(\tilde{H})|,$$

where $\tilde{H}$ is a full covering group of H. Hence,

$$\mathscr{X}(H) \cong T(\tilde{H})$$

is obtained.

(iii) Let $p = 2$. Also let $\phi\colon H \to H/H^2$ be the natural epimorphism, and $\hat{\phi}\colon \mathscr{X}(H) \to \mathscr{X}(H/H^2)$ be the extension of ϕ, obtained as in Proposition 4.1.13. Then,

$$\begin{aligned} \ker \hat{\phi} &= \langle H^2, H^{2\psi} \rangle [H^2, H^{\psi}] \\ &= \langle H^2, H^{2\psi} \rangle\, D^2. \end{aligned}$$

Also, by Proposition 4.1.12,

$$\langle H^2, H^{2\psi}\rangle \cap D = [H^2, H^{2\psi}] = D^4.$$

Thus we have that

$$|\mathscr{X}(H)| = |\mathscr{X}(H/H^2)|\, |\langle H^2, H^{2\psi}\rangle| \frac{|D^2|}{|D^4|},$$

$$|\mathscr{X}(H)| \text{ divides } |\mathscr{X}(H/H^2)|\, |\mathscr{X}(H^2)| \frac{|D(H)^2|}{|D(H)^4|}.$$

The rest of the theorem is easy to verify.

EXAMPLE 4.2.5. The following example shows that the action of $\mathscr{X}(H)$ on $W(H)$ need not be nilpotent.

Let p be an odd prime, A an elementary abelian p-group of order p^2 generated by a_1 and a_2. Then by the previous theorem,

$$W(A) = \langle [a_1, a_2^{\psi}]\rangle \subseteq Z(\mathscr{X}(A)).$$

The automorphism b of A defined by $b\colon a \to a^{-1}$ for all $a \in A$, may be extended to an automorphism of $\mathscr{X}(A)$ by

$$b\colon a \to a^{-1}, \qquad a^{\psi} \to a^{\psi} \qquad \text{for all } a \in A.$$

Let b^{ψ} be the automorphism of $\mathscr{X}(A)$ determined by

$$b^{\psi}\colon a \to a, \qquad a^{\psi} \to a^{-\psi} \qquad \text{for all } a \in A.$$

Denote $\langle b\rangle$ by B. Then, $\mathscr{X}(B)$ is an elementary abelian group of order 4, and $\mathscr{X}(B) \subseteq \mathrm{Aut}(\mathscr{X}(A))$.

Let $H = AB$, $G = \mathscr{X}(A)\,\mathscr{X}(B)$ be the semi-direct product groups constructed with respect to the actions defined above. Then, $G = \langle H, H^{\psi}\rangle$ and, as can be easily checked, $hh^{\psi} = h^{\psi}h$ for all $h \in H$. Thus, there exists an epimorphism $\lambda\colon \mathscr{X}(H) \to G$ such that $\lambda\colon a \to a$, $a^{\psi} \to a^{\psi}$, $b \to b$, $b^{\psi} \to b^{\psi}$. Now, $z = [a_1, a_2^{\psi}] \in W(H)$, and $z \neq 1$ since $z^{\lambda} = [a_1, a_2^{\psi}] \neq 1$ in $\mathscr{X}(A)$. Also, $\lambda\colon z^b \to [a_1, a_2^{\psi}]^b = [a_1^{-1}, a_2^{\psi}] = [a_1, a_2^{\psi}]^{-1} = z^{-1}$ in G. Hence, $[z, b, b, \ldots] \neq 1$.

4.3. *Bounds for Exponents and Orders*

Here we let H be a finite group and derive upper bounds for the exponents of $W(H)$, $\mathbf{M}(H)$, $D(H)$, and for $|\mathbf{M}(H)|$. The bounds for $\exp(\mathbf{M}(H))$ and $|\mathbf{M}(H)|$ in the following theorem are similar to those obtained by W. R. Jones in [8] for $M(H)$.

THEOREM 4.3.1. *Let H be a finite group, and let Z be a subgroup of $Z(H) \cap H'$. Then,*

(i) $\exp(W(H)) \mid \exp(W(H/Z)) \exp(Z)$,

(ii) $\exp(\mathbf{M}(H)) \mid \exp(\mathbf{M}(H/Z)) \exp(Z \otimes H/H'Z(H))$,

(iii) $|Z| \, |\mathbf{M}(H)| \mid |\mathbf{M}(H/Z)| \, |Z \otimes H/H'Z(H)|$.

Proof. Let $K = H/Z$ in Proposition 4.1.13. Then we have the diagram below, where $[Z, H^\psi] \subseteq W(H)$ is justified by Lemma 4.1.9, and

$$[H', Z(H)^\psi] \subseteq R(H) \subseteq W(H)$$

is justified by Lemma 4.1.11.

$$\begin{tikzcd}
& D(H) \arrow[r, "\hat{\phi}_D"] \arrow[d, dash, "H'/Z"] & D(K) \arrow[d, dash, "K'"] \\
& D(H) \cap L(H)Z \arrow[r] \arrow[d, dash, "Z"] & W(K) \arrow[dd, dash] \\
& W(H) \arrow[d, dash] & \\
& R(H)[Z, H^\psi] \arrow[r] \arrow[dl, dash] \arrow[d, dash] & R(K) \arrow[d, dash] \\
R(H) \arrow[d, dash] & {[Z, H^\psi]} \arrow[r] & 1 \\
{[H', Z(H)^\psi]} & &
\end{tikzcd}$$

By Lemma 4.1.9,

$$\exp([Z, H^\psi]) \mid \exp(Z);$$

thus it follows from the diagram that

$$\exp(W(H)) \mid \exp(W(K)) \exp(Z);$$

so part (i) is proved.

Since $Z \subseteq H' \cap Z(H)$ and $[H', Z(H)^\psi] \subseteq R(H)$, we have that

$$[Z, Z(H)^\psi], [H', Z^\psi] \qquad (=[Z, H'^\psi])$$

are subgroups of $R(H)$. Therefore,

$$[Z, (Z(H)\, H')^{\psi}] = 1 \text{ modulo } R(H),$$

from which it follows that $R(H)[Z, H^{\psi}]/R(H)$ is a homomorphic image of $Z \otimes (H/Z(H)\, H')$. Now part (ii) follows directly from the diagram.

As for part (iii), it is derived from

$$|\mathbf{M}(H)| = \left|\frac{W(H)}{R(H)}\right| = \left|\frac{W(H)}{R(H)[Z, H^{\psi}]}\right| \left|\frac{R(H)[Z, H^{\psi}]}{R(H)}\right|,$$

and

$$\left|\frac{W(H)}{R(H)[Z, H^{\psi}]}\right| = \frac{1}{|Z|}\left|\frac{W(K)}{R(K)}\right| = \frac{1}{|Z|}\, |\mathbf{M}(K)|.$$

THEOREM 4.3.2. *Let H be a finite nilpotent group, $d(H)$ be the minimum number of generators of H, and $d_1(H) = d(H/Z(H)\, H')$. Then,*

$$|\mathbf{M}(H)| \mid |M(H/H')|\, |H'|^{d_1(H)-1}.$$

Proof. Let Z be a subgroup of $Z(H) \cap H'$. Clearly,

$$|Z \otimes H/H'Z(H)| \text{ divides } |Z|^{d_1(H)}.$$

Thus, by (iii) of the previous theorem, we have

$$|\mathbf{M}(H)| \text{ divides } |\mathbf{M}(H/Z)|\, |Z|^{d_1(H)-1}.$$

Let $\bar{H} = H/Z$. Then it is easy to see that $d_1(\bar{H}) \leqslant d_1(H)$. By induction on the nilpotency class of H, we reach

$$|\mathbf{M}(H)| \text{ divides } |\mathbf{M}(H/H')|\, |H'|^{d_1(H)-1}.$$

The proof is now concluded, since by Theorem 4.2.4, $\mathbf{M}(H/H') \cong M(H/H')$.

As a direct consequence of this result we obtain the Gaschutz–Vermani theorem [10].

COROLLARY 4.3.3. *Let H be a finite nilpotent group. Then,*

$$|M(H)| \mid |M(H/H')|\, |H'|^{d_1(H)-1}.$$

LEMMA 4.3.4. *Let H be a finite group, $\tilde{H}$ be a central extension of H, and Z a subgroup of $Z(\tilde{H})$ such that $\tilde{H}/Z \cong H$. Then,*

$$\tilde{H}' \cap Z \textit{ is a homomorphic image of } M(H).$$

Proof. Let H be defined by

$$\{h_1, \ldots, h_n \mid \omega_j(h_1, \ldots, h_n) = 1,\ 1 \leqslant j \leqslant m\}.$$

Also, let

$$1 \longrightarrow Z \xrightarrow{\text{id.}} \tilde{H} \xrightarrow{u} H \longrightarrow 1$$

be an exact sequence, and define

$$\hat{H} = \langle \hat{h}_i \mid u(\hat{h}_i) = h_i,\ 1 \leqslant i \leqslant n\rangle.$$

Then, $\tilde{H} = \hat{H}Z$, $\tilde{H}' = \hat{H}'$.

Define $z_j = \omega_j(\hat{h}_i)$ for $1 \leqslant j \leqslant m$, and let $Z_0 = \langle z_1, \ldots, z_m\rangle$. Since $\hat{H}/Z_0 \cong H$, we have $\hat{H} \cap Z = Z_0$. Hence,

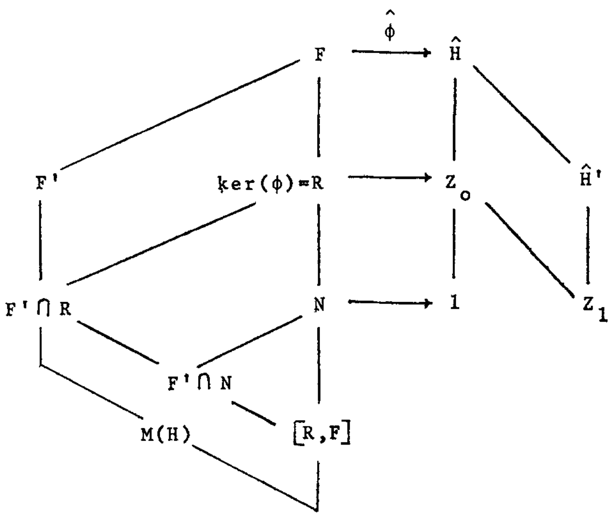
$$\tilde{H}' \cap Z = \hat{H}' \cap Z = \hat{H}' \cap Z_0.$$

Let $Z_1 = \hat{H}' \cap Z_0$. We will prove that Z_1 is a homomorphic image of $M(H)$. To this end we let $F = \langle x_1, \ldots, x_n\rangle$ be a free group, $\phi: F \to H$, where $\phi(x_i) = h_i$, $1 \leqslant i \leqslant n$, be a presentation for H, and $R = \ker(\phi)$.

We note that $\hat{H}$ has the presentation $\hat{\phi}: F \to \hat{H}$ which is determined by

$$\hat{\phi}(x_i) = \hat{h}_i, \qquad 1 \leqslant i \leqslant n.$$

We have the following diagram



Now let $x \in Z_1$ $(=\hat{H}' \cap Z_0)$. As $\hat{\phi}(F') = \hat{H}'$, and $\hat{\phi}(R) = Z_0$, there exist $f' \in F'$ and $r \in R$ such that $x = \hat{\phi}(f') = \hat{\phi}(r)$.

Thus,

$$f'r^{-1} \in \ker(\hat{\phi})(=N), \qquad f' \in NR(=R),$$

$f' \in F' \cap R$. Hence, $\hat{\phi}(F' \cap R) = Z_1$.

The rest of the proof is clear; for $Z_1 \cong (F' \cap R)/(F' \cap R \cap N) = (F' \cap R)/(F' \cap N)$ which is a homomorphic image of $((F' \cap R)/[R, F])$ $(\cong M(H))$.

Remark 4.3.5. Suppose H is a group which is not necessarily finite, yet is finitely presented and Hopfian. Let $\tilde{H}$ be a central extension of H and let Z be a subgroup of $\tilde{H}' \cap Z(H)$ such that $\tilde{H}/Z = H$. Then the argument in the first paragraph of the previous lemma shows that $\tilde{H} = \hat{H}$ and $Z = Z_0$, and thus in particular Z is finitely generated.

Our next result gives an upper bound for $\exp(D(H))$ which involves the derived structure of H.

THEOREM 4.3.6. *Let H be a finite group. Then,*

$$\exp(D(H)) \mid \exp(M(H/H'))\exp(M(H'))\exp(H').$$

Proof. Since $DH' = DL_1$ and $[D, L_1] = 1$, we have that $[D, H'] \subseteq D'$. Hence, as $[H^{\psi}, H'] \subseteq D$, we obtain $[H^{\psi}, H', H'] \subseteq D'$. Thus,

$$[H^{\psi}, H']^{\exp(H')} \subseteq D'. \tag{1}$$

We also have

$$[H^{\psi}, H']^{\exp[H,H']} \subseteq D \cap L, \tag{2}$$

which follows from the fact

$$[H^{\psi}, H'] = [H, H'] \text{ modulo } L.$$

Hence, from (1) and (2) we get

$$[H^{\psi}, H']^{\exp(H')} \subseteq D(H)' \cap W(H). \tag{3}$$

Let $N = \langle H', H'^{\psi}\rangle^{\mathfrak{X}(H)}$. Then, by Propositions 4.1.12, 4.1.13,

$$\frac{D(H)N}{N} \cong D\left(\frac{H}{H'}\right), \tag{4}$$

$$D(H) \cap N = [H, H'^{\psi}]. \tag{5}$$

Also, by Theorem 4.2.1, we have

$$D\left(\frac{H}{H'}\right) = W\left(\frac{H}{H'}\right). \tag{6}$$

Therefore, by (4) and (6),

$$(D(H)N)^{\exp(W(H/H'))} \subseteq N. \tag{7}$$

Thus, by (7) and (5),

$$D(H)^{\exp(W(H/H'))} \subseteq D(H) \cap N = [H, H'^{\psi}]. \tag{8}$$

Hence, by (8) and (3),

$$D(H)^{\exp(W(H/H'))\exp(H')} \subseteq D(H)' \cap W(H). \tag{9}$$

Since by Lemma 4.3.4, $D(H)' \cap W(H)$ is a homomorphic image of $M(D(H)/W(H))$ which is isomorphic to $M(H')$, we get from Proposition 4.1.4 that

$$\exp(D(H)) \mid \exp(W(H/H')) \exp(H') \exp(M(H')). \tag{10}$$

We give some direct consequences of the theorem.

COROLLARY 4.3.7. *Let H be a finite group. Then,*

$$\exp(M(H)) \mid \exp(M(H/H')) \exp(M(H')) \exp(H').$$

Proof. It suffices to note that $M(H)$ is a section of $D(H)$, and thus,

$$\exp(M(H)) \mid \exp(D(H)).$$

COROLLARY 4.3.8. *Let H be a finite group. Then,*

(i) $\exp(\mathscr{X}(H)) \mid \exp(H) \exp(H') \exp(M(H/H')) \exp(M(H'))$,
(ii) $\pi(\mathscr{X}(H)) = \pi(H)$.

Proof. Part (i) follows from $\mathscr{X}(H)^{\exp(H)} \subseteq D(H)$, and from the previous theorem. Part (ii) follows from the well-known result $M(H)^{|H|} = 1$.

COROLLARY 4.3.9. *Let H be a finite nilpotent group. Then $\mathscr{X}(H)$ is also finite nilpotent.*

Proof. This follows from Corollary 4.1.15, and from $\pi(\mathscr{X}(H)) = \pi(H)$. Given the two corollaries above, Theorem C is now fully established.

4.4. *H Perfect*

The purpose of this subsection is to prove the following theorem and its corollary.

THEOREM 4.4.1. *Let H be a perfect group. Then there exists $\tilde{H}$ a covering group of H such that $\mathscr{X}(H) \cong T(\tilde{H})$.*

COROLLARY 4.4.2. (i) *The group $\tilde{H}$ in Theorem* 4.4.1 *is a maximal stem extension of G. Also, the maximal stem extensions of G are all isomorphic.*

(ii) *Every automorphism of H lifts up to a unique automorphism of $\tilde{H}$.*

We note that part (i) of the corollary is a result of Schur (see [6], p. 214). As for part (ii), a proof of it was given by J. Alperin (see [5], p. 356).

Throughout the remainder of this subsection, H will be assumed to be a perfect group.

LEMMA 4.4.4. $T(H) = H \times H \times H$.

Proof. This is but Remark 2.3.1, part (iv).

LEMMA 4.4.5. *Let σ be an automorphism of H such that $[h, \sigma(h)] = 1$ for all $h \in H$. Then, $\sigma = 1$.*

Proof. As $\sigma(h)$ commutes with h for all $h \in H$, we have as an application of Proposition 4.1.7, part (ii), that

$$[H, H^{\sigma}] \text{ centralizes } [H, \sigma].$$

Thus, as $[H, H^{\sigma}] = [H, H] = H$, we get $[H, \sigma] \subseteq Z(H)$. Now, considering that $[\sigma, H, H]$, and $[H, \sigma, H]$ are trivial groups; we conclude that $[H, H, \sigma] = [H, \sigma] = 1$.

LEMMA 4.4.6. *The following facts holds in $\mathscr{X}(H)$.*

(i) $\mathscr{X} = DL$, $W \leqslant Z(\mathscr{X})$,

(ii) *D and $\mathscr{X}$ are perfect groups,*

(iii) $DL_1 = DH = L_1H$, $L_1 \cap L_2 = W = D \cap L_1$,

(iv) $L = L_1L_2$.

Proof. (i) Since $\mathscr{X}/DL \simeq H/H'$, and as $H = H'$, we have $\mathscr{X} = DL$. Thus, by Proposition 4.1.7, part (ii), $W \subseteq Z(\mathscr{X})$.

(ii) Since $D/W \simeq H' = H$, $D = D'W$ follows. Let $\bar{\mathscr{X}} = \mathscr{X}/D'$. Then, $\bar{D}$ $(=\overline{[H^{\psi}, H]}) = \bar{W}$ which is central in $\bar{\mathscr{X}}$. Therefore,

$$[\overline{H^{\psi}}, \bar{H}, \overline{H^{\psi}}] = 1 = [\bar{H}, \overline{H^{\psi}}, \overline{H^{\psi}}],$$

and so,

$$[\overline{H^{\psi}}, \overline{H^{\psi}}, \bar{H}] = [\overline{H^{\psi}}, \bar{H}] = 1;$$

that is, the conclusion $D = D'$ is reached.

Since $\mathscr{X}/W \simeq H \times H \times H$ is a perfect group, and since

$$W \subseteq D = [H, H^{\psi}] \subseteq \mathscr{X}',$$

we have

$$\mathscr{X} = \mathscr{X}'W = \mathscr{X}'.$$

(iii) The assertion $DL_1 = DH = L_1H$ follows from Lemma 4.1.10, part (iv). As for $L_1 \cap L_2 = W$, it is part (iii) of the same lemma. Since

$$D \cap L_1 = D \cap L \cap L_1 = W \cap L_1 = W,$$

we are done with this part.

(iv) Since $DH = DL_1$ and $DH^{\psi} = DL_2$, we have

$$\mathscr{X} = DH \cdot DH^{\psi} = DL_1DL_2 = DL_1L_2.$$

Thus, $L = (L \cap D)L_1L_2 = WL_1L_2 = L_1L_2$ follows.

PROPOSITION 4.4.7. *Suppose a group A contains a pair of subgroups A_1, A_2 and a third subgroup K such that*

(i) $A = A_1A_2$, $[A_1, A_2] = 1$, $A_1 \cap A_2 = Z$,

(ii) $A = A_1K = A_2K$, $A_1 \cap K = A_2 \cap K = 1$.

Then, $A_1 \cong A_2$.

Proof. There exist isomorphism σ_1', σ_2' from A_1/Z, A_2/Z onto K, respectively. For $i = 1, 2$, define σ_i: $K \to A_i$ such that $\sigma_i(1) = 1$ and $\sigma_i'\sigma_i(k) = k$ for all $k \in K$. Also, define $\mathscr{U}_i = \sigma_i(K)$ for $i = 1, 2$. Since $\mathscr{U}_1\mathscr{U}_2$ is a transversal for Z in A, there exist σ_1: $K \to \mathscr{U}_1$, σ_2: $K \to \mathscr{U}_2$, ζ: $K \to Z$ functions such that $k = \zeta(k)\, k^{\sigma_1}k^{\sigma_2}$ for all $k \in K$.

Define for $i = 1, 2$, ϕ_i: $K \times K \to Z$ by

$$\phi_i(k_1, k_2) = (k_1^{\sigma_i}k_2^{\sigma_i})((k_1k_2)^{\sigma_i})^{-1}$$

for all $k_1, k_2 \in K$. Then, ϕ_1, ϕ_2 are normalized 2-cocycles, elements of $Z^2(K, Z)$.

Since for any $k_1, k_2 \in K$,

$$\begin{aligned} k_1k_2 &= (\zeta(k_1)\, k_1^{\sigma_1}k_1^{\sigma_2})(\zeta(k_2)\, k_2^{\sigma_1}k_2^{\sigma_2}), \\ k_1k_2 &= \zeta(k_1)\, \zeta(k_2)\, \phi_1(k_1, k_2)\, \phi_2(k_1, k_2)(k_1k_2)^{\sigma_1}(k_1k_2)^{\sigma_2} \\ &= \zeta(k_1k_2)(k_1k_2)^{\sigma_1}(k_1k_2)^{\sigma_2}, \end{aligned}$$

we have that $\phi_1\phi_2 = \hat{\zeta}$, where $\hat{\zeta}$ is a 2-coboundary such that

$$\hat{\zeta}(k_1, k_2) = \zeta(k_1k_2)\, \zeta(k_1)^{-1}\, \zeta(k_2)^{-1}.$$

Let γ: $A_1 \to A_2$ be defined by

$$\gamma(zk^{\sigma_1}) = z^{-1}\zeta(k)k^{\sigma_2}$$

for all $k \in K$, $z \in Z$. Then it can be checked directly that γ is an isomorphism from A_1 onto A_2.

Proof of Theorem 4.4.1. Let $\bar{\mathscr{X}} = \mathscr{X}/W(H)$, $Z = W(H)$, and let π_1, π_2, π_3 be the projections of $\bar{\mathscr{X}}$ on $\bar{L}_1$, $\bar{D}$ and $\bar{L}_2$ respectively. Furthermore, let $\bar{\sigma}_1$ and $\bar{\sigma}_2$ be the respective restrictions of π_1 and π_2 to $\bar{H}$, and let $\bar{\tau}_1$ and $\bar{\tau}_2$ be the respective restrictions of π_2 and π_3 to $\bar{H}^{\bar{\psi}}$. Then the maps $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are isomorphisms from $\bar{H}$ onto $\bar{L}_1$ and onto $\bar{D}$ respectively.

We claim that $\bar{\psi}\bar{\tau}_1 = \bar{\sigma}_2$. Let $\bar{h} \in \bar{H}$. Then since $H \subseteq L_1 D$, we have that

$$\bar{h} = \bar{h}^{\bar{\sigma}_1}\bar{h}^{\bar{\sigma}_2}, \qquad \bar{h}^{\bar{\psi}} = \bar{h}^{\bar{\psi}\bar{\tau}_1}\bar{h}^{\bar{\psi}\bar{\tau}_2},$$

$$1 = [\bar{h}, \bar{h}^{\bar{\psi}}] = [\bar{h}^{\bar{\sigma}_2}, \bar{h}^{\bar{\psi}\bar{\tau}_1}],$$

and $1 = [\bar{d}, \bar{d}^{\bar{\sigma}_2^{-1}\bar{\psi}\bar{\tau}_1}]$ for $\bar{d} = \bar{h}^{\bar{\sigma}_2}$. Let $\omega = \bar{\sigma}_2^{-1}\bar{\psi}\bar{\tau}_1$. Then ω is an automorphism of $\bar{D}$ and $\bar{d}$ commutes with $\bar{d}^{\omega}$ for any $\bar{d} \in \bar{D}$. Hence, by Lemma 4.4.5, $\omega = 1$.

Proposition 4.4.7 is applicable to the pairs (L_1, D) and (D, L_2). For every $\bar{h} \in \bar{H}$, we make the following choices:

$$h^{\sigma_1} \in \bar{h}^{\bar{\sigma}_1}, \qquad h^{\sigma_2} \in \bar{h}^{\bar{\sigma}_2}, \qquad h^{\psi\tau_1}(= k^{\sigma_2}) \in \bar{h}^{\bar{\psi}\bar{\tau}_1}, \qquad \text{and} \qquad h^{\psi\tau_2} \in \bar{h}^{\bar{\psi}\bar{\tau}_2}—$$

in all cases, we choose $1 \in \bar{1}$—and so we have the following maps,

$$\sigma_1 \colon H \to L_1, \qquad \sigma_2 \colon H \to D, \qquad \tau_1 \colon H^{\psi} \to D, \qquad \tau_2 \colon H^{\psi} \to L_2 .$$

Furthermore, we have $\zeta_1, \zeta_2 \colon H \to Z$, $\phi_1, \phi_2 \in C^2(H, Z)$, and $\phi_2', \phi_3' \in C^2(H^{\psi}, Z)$, functions which satisfy

$$h = \zeta_1(h)\, h^{\sigma_1}h^{\sigma_2}, \qquad h^{\psi} = \zeta_2(h)\, h^{\psi\tau_1}h^{\psi\tau_2},$$

$$\phi_i(h_1, h_2) = h_1^{\sigma_i}h_2^{\sigma_i}((h_1h_2)^{\sigma_i})^{-1} \qquad \text{for} \quad i = 1, 2,$$

$$\phi_i'(h_1, h_2) = h_1^{\psi\tau_i}h_2^{\psi\tau_i}((h_1h_2)^{\psi\tau_i})^{-1} \qquad \text{for} \quad i = 2, 3,$$

and for all $h, h_1, h_2 \in H$.

We observe that $\phi_1, \phi_2, \phi_2', \phi_1'$ are normalized 2-cocycles and that

$$\phi_2(h_1, h_2) = \phi_2'(h_1^{\psi}, h_2^{\psi})$$

for all $h_1, h_2 \in H$.

By the proof of Proposition 4.4.7, the functions

$$\gamma_1 \colon L_1 \to D, \qquad \gamma_2 \colon D \to L_2$$

which are defined by

$$\gamma_1(zh^{\sigma_1}) = z^{-1}\zeta_1(h)h^{\sigma_2}, \qquad \gamma_2(zh^{\sigma_2}) = z^{-1}\zeta_2(h)h^{\psi\tau_2},$$

for all $z \in Z$, $h \in H$, are isomorphisms from L_1 onto D, and from D onto L_2, respectively. Therefore, the function $\gamma_3: L_1 \to L_2$ defined by $\gamma_3 = \gamma_2\gamma_1$ is an isomorphism from L_1 onto L_2. Hence, the function $\gamma: L_1 \times L_1 \times L_1 \to \mathscr{X}(H)$, defined by $\gamma(\omega_1, \omega_2, \omega_3) = \omega_1\gamma_1(\omega_2)\,\gamma_3(\omega_3)$ is an epimorphism and

$$\ker(\gamma) = \{(z_1, z_1z_2, z_2) \mid z_1, z_2 \in Z\}.$$

Proof of Corollary 4.4.2. Let $(Z \mid \hat{H})$ be a maximal stem extension of H. We may assume, $\hat{H}/Z = H$. By considering the epimorphism $\lambda_{\hat{H}}: \mathscr{X}(H) \to T(\hat{H})$, we conclude that $\lambda_{\hat{H}}(L_1(H)) = \overline{\hat{H} \times 1 \times 1}$ and $\lambda_{\hat{H}}(W(H)) = \overline{1 \times Z \times 1}$. Thus, $\lambda_{\hat{H}}$ may be considered as an epimorphism from $(W \mid L_1)$ into $(Z \mid \hat{H})$. Hence, as $(Z \mid \hat{H})$ is maximal, by definition, $\lambda_{\hat{H}}$ is an isomorphism, and part (i) is proved.

Now let $\alpha \in \mathrm{Aut}(H)$ and define $\tilde{\alpha}: H \cup H^{\psi} \to H \cup H^{\psi}$ by $\tilde{\alpha}(h) = \alpha(h)$, $\tilde{\alpha}(h^{\psi}) = \alpha(h)^{\psi}$. Then the map $\tilde{\alpha}$ extends naturally to an automorphism of $\mathscr{X}(H)$, which leaves $D(H)$ $(\simeq \tilde{H})$ and $W(H)$ invariant. Suppose $\tilde{\alpha}$ induces the trivial automorphism on $D(H)$. Then, as $\mathscr{X} = LD$, we get that $[\mathscr{X}, \tilde{\alpha}] \subseteq L$. However, as $[H, \tilde{\alpha}] = [H, \alpha]$ is a subgroup of H, and as $H \cap L = 1$, we arrive at $\alpha = 1$. Now, the uniqueness of the extension of α to an automorphism of $\tilde{H}$ follows from the fact that central automorphisms of perfect groups are necessarily trivial.

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